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On the complexity of computing the L_q norm

We study the complexity of computing

$$S(f) \quad = \quad \|f\|_{L_q(Q)} = \begin{cases} \left(\int_Q |f(x)|^q dx \right)^{1/q} & \text{if} \quad 1 \le q < \infty \\ \operatorname{ess\,sup}_{t \in Q} |f(t)| & \text{if} \quad q = \infty, \end{cases}$$

that is, the L_q norm of a function. We assume that f is from the unit ball of the Sobolev space $W_p^r(Q)$, where $Q = [0,1]^d$, and $r \in \mathbf{N}$, $1 \le p, q \le \infty$ are such that $W_p^r(Q)$ is embedded into $L_q(Q)$. Information is standard, that is, consists of function values. Thus, we want to determine the (order of the) optimal error $e_n(S)$ of approximating the norm $S(f)$, based on algorithms using not more than n values of f.
A general result of G. W. Wasilkowski (Some nonlinear problems are as easy as the approximation problem, Comp. & Maths. with Appls. 10 (1984), 351-363) states that in the deterministic setting the n-th minimal errors $e_n^{\rm det}(S)$ are of the same order as $e_n^{\rm det}(J)$, where J is the embedding $J : W_p^r(Q) \to L_q(Q)$, thus we can derive the order of $e_n^{\rm det}(S)$ directly from known results on approximation. In the randomized setting such a general result no longer holds.
We present and analyze a randomized algorithm for computing $S(f)$ and provide lower bounds, which allow to determine the order of the randomized n-th minimal errors $e_n^{\rm ran}(S)$. We also provide comparisons to $e_n^{\rm det}(S)$ as well as to $e_n^{\rm ran}(J)$ and discuss some extensions and open problems.