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On a generalized nonlinear heat equation in Besov and Triebel-Lizorkin spaces

We deal with a generalized nonlinear heat equation

$$\frac{\partial}{\partial t} u(x, t) + (-\Delta_x)^\alpha u(x, t) = \sum_{j=1}^n \frac{\partial}{\partial x_j} u^2(x, t), \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (1)$$

$$u(\cdot, 0) = u_0(x), \quad x \in \mathbb{R}^n \quad (2)$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $u(x, t)$ is a scalar function. The case $\alpha = 1$ corresponds to a classical nonlinear heat equation.

In our approach we consider solutions which solve the fixed point problem $T_{u_0} u = u$ whereas the operator T_{u_0} is given as

$$T_{u_0} u(x, t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha Du^2(x, \tau) d\tau, \quad x \in \mathbb{R}^n, \quad 0 < t < T$$

in some weighted Lebesgue spaces $L_v((0, T), b, X)$. Here $X = A_{p,q}^s(\mathbb{R}^n)$ with $A \in \{B, F\}$ is a Banach space of Besov or Triebel-Lizorkin type and $W_t^\alpha \omega$ is defined as

$$W_t^\alpha \omega(x) := \left[\frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}} \right)^\vee * \omega \right] (x), \quad t > 0, \quad x \in \mathbb{R}^n$$

for $\omega \in S'(\mathbb{R}^n)$. For this purpose we need the following apriori estimate

$$t^{d/2\alpha} \|W_t^\alpha \omega\|_{A_{p,q}^{s+d}(\mathbb{R}^n)} \leq c \|\omega\|_{A_{p,q}^s(\mathbb{R}^n)}, \quad 0 < t \leq 1 \quad (3)$$

with $1 \leq p, q \leq \infty$ ($p < \infty$ in case of F -spaces), $s \in \mathbb{R}$, $d \geq 0$ and $\alpha \in \mathbb{N}$. To prove (3) we use decomposition methods of the spaces $A_{p,q}^s(\mathbb{R}^n)$ by means of wavelets and molecules. Because of the structure of the nonlinearity these spaces have to fulfill certain multiplication properties (multiplication algebra, Hölder inequalities). The initial data u_0 belong to some spaces $A_{p,q}^\sigma(\mathbb{R}^n)$ with $s \geq \sigma > s - \alpha$ such that we are in the supercritical case. We ask for solutions of (1), (2) which are additionally strong and stable such that under our conditions the problem is well-posed.