The Gaussian Capacities

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Outline

1. Motivation

Sobolev and isoperimetric inequalities

Maz'ya in 1960s proved the following are equivalent:

- Sobolev inequality: for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$\Big(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}} \lesssim \int_{\mathbb{R}^n} |\nabla f|\, dx; \quad (\dot{W}^{1,1}(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n))$$

- Isoperimetric inequality: for smooth domains E in $\mathbb{R}^n$,

$$|E|^{\frac{n-1}{n}} \lesssim \mathcal{H}^{n-1}(\partial E).$$

Here $\mathcal{H}^{n-1}$ means the $(n-1)$-dimensional Hausdorff measure.

- [M60] **V. Maz'ya**, Dokl. Akad. Nauk SSSR 133 (1960), 527-530 (Russian).
- [M61] **V. Maz'ya**, Dokl. Akad. Nauk SSSR 140 (1961), 299-302. (Russian).

More general, for open $\Omega \subset \mathbb{R}^n$ and $q \in [1, \infty)$, the following are equivalent:

- Sobolev inequality: for every $f \in C_c^\infty(\Omega)$,

$$\Big(\int_\Omega |f|^q \, d\mu \Big)^{1/q} \lesssim \int_\Omega |\nabla f| \, dx; \quad (\dot{W}^{1,1}(\Omega) \subset L^q(\Omega, \mu))$$

- Isoperimetric inequality: for every bounded open set E with smooth boundary, $\overline{E} \subset \Omega$,

$$[\mu(E)]^{1/q} \lesssim \mathcal{H}^{n-1}(\partial E).$$

• [M03] **V. Maz'ya**, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 307 – 340, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003

When the gradient is integrable to a power> 1, the isoperimetric inequality has to be replaced with an isocapacitary inequality: for open $\Omega \subset \mathbb{R}^n$ and $q \geq p \geq 1$, the following are equivalent:

- Sobolev inequality: for every $f \in C_c^\infty(\Omega)$,

$$\Big(\int_\Omega |f|^q \, d\mu \Big)^{1/q} \lesssim \|\nabla f\|_{L^p(\Omega)}; \quad (\dot{W}^{1,p}(\mathbb{R}^n) \subset L^q(\Omega, \mu))$$

- Isocapacitary inequality: for every bounded open set E with smooth boundary, $\overline{E} \subset \Omega$,

$$[\mu(E)]^{p/q} \lesssim \mathrm{cap}_p(\overline{E}),$$

  with Sobolev p-capacity $\mathrm{cap}_p(A) := \inf_{C_c^\infty(\Omega) \ni u \geq 1 \text{ on } A} \int_\Omega |\nabla u|^p \, dx$.

- [M85] **V. Maz'ya**, Sobolev Spaces, Springer-Verlag, 1985.

Capacity

- Capacity: A real valued function C defined on all subsets of a metric space $\mathcal{X}$ is called a capacity if it is

  1. (non-negative): $C(E) \geq 0$, for all $E \subset \mathcal{X}$.
  2. (monotonic): If $E_1 \subset E_2 \subset \mathcal{X}$, then $C(E_1) \leq C(E_2)$.
  3. (countably subadditive): For any sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of $\mathcal{X}$,

$$C\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} C(E_j).$$

• The notion of capacity is originated from physics (electrostatics), and nowadays widely used in analysis, geometry and mathematical physics.

Generalizations

• The equivalence between the Sobolev inequalities and the isoperimetric-isocapacitary inequality have be generalized to many other settings, including:

$*$ Weighted Euclidean spaces (Turesson 00)

$*$ Riemannian manifolds (Maz'ya 03, ...)

$*$ graph (Maz'ya 03, ...)

$*$ metric spaces with doubling measures (Shanmugalingam 00, Kinnunen-Korte 08, ...)

• Applications: PDEs, Potential Analysis,...

• All above need doubling measure! How about non-doubling cases?

Doubling measure:

A measure μ on a metric space X is called doubling, if

$$\mu(B(x,2r)) \leq C\mu(B(x,r))$$

for all $x \in X$ and $r \in (0, \frac{\operatorname{diam} X}{2})$.

• The notion of doubling measures was introduced by Coifman and Weiss [CW71,CW77] and known as a basic assumption for many classical theory of harmonic analysis.

• [CW71] **R. Coifman and G. Weiss**, Lecture Notes in Math. 242, Springer-Verlag, Berlin-New York, 1971.

• [CW77] **R. Coifman and G. Weiss**, Bull. Amer. Math. Soc. 83 (1977), 569-645.

Gaussian Spaces

- $\mathbb{G}^n := (\mathbb{R}^n, dV_\gamma)$ — the Gaussian space
  - $dV_\gamma(x) := \gamma(x)dx := (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$ — the Gaussian measure
  - arises from probability theory, quantum mechanics, ...
  - a typical non-doubling probability measure:

$$V_\gamma(\mathbb{R}^n) = \int_{\mathbb{R}^n} \gamma(x)\, dx = 1.$$

- $C_c(\mathbb{R}^n)$ — the class of continuous functions with compact support in $\mathbb{R}^n$

  $C_c^k(\mathbb{R}^n)$ — all k-times continuously differentiable functions with compact support in $\mathbb{R}^n$

Gaussian Poincaré Inequality

- One key property of Gaussian space is the Gaussian Poincaré Inequality: $\forall\, f \in C_c^1(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n}\left|f-\int_{\mathbb{R}^n} f\,dV_\gamma\right|^p dV_\gamma\right)^{1/p} \le C\left(\int_{\mathbb{R}^n}|\nabla f|^p\,dV_\gamma\right)^{1/p}.$$

- $p \in [1,\infty)$: [P86] **G. Pisier**, Lecture Notes in Math. 1206, Springer, Berlin, 1986, 167-241.
- Sharp constant:
  - $p = 1$: [L96] **M. Ledoux**, Lecture Notes in Math. 1648, Springer, Berlin, 1996, 165-264.
  - $p = 2$: [CMN10] **V. Caselles, M. Jr. Miranda and M. Novaga**, J. Funct. Anal. 259 (2010), 1491-1516.
  - $p \ge 2$: [Z14] **Q. Zeng**, J. Funct. Anal. 266 (2014), 3236-3264.

• An equivalent statement of the Gaussian Poincaré Inequality is as follows: $\forall\, f \in C_c^1(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |f|^p \, dV_\gamma\right)^{1/p} \leq C\left[\left(\int_{\mathbb{R}^n} |\nabla f|^p \, dV_\gamma\right)^{1/p} + \left|\int_{\mathbb{R}^n} f \, dV_\gamma\right|\right].$$

• Write

$$\|f\|_{W^{1,p}(\mathbb{G}^n)} := \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dV_\gamma\right)^{1/p} + \left|\int_{\mathbb{R}^n} f \, dV_\gamma\right|.$$

Then the above Sobolev type inequality is equivalent to the Sobolev embedding

$$W^{1,p}(\mathbb{G}^n) \subset L^p(V_\gamma).$$

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Question: given a non-negative Borel measure μ on $\mathbb{R}^n$ and $q \in (0, \infty)$, when we have the embedding

$$W^{1,p}(\mathbb{G}^n) \subset L^q(\mu)?$$

More precise, when

$$\left(\int_{\mathbb{R}^n} |f|^q \, d\mu\right)^{1/q} \le C\left(\left(\int_{\mathbb{R}^n} |\nabla f|^p \, dV_\gamma\right)^{1/p} + \left|\int_{\mathbb{R}^n} f \, dV_\gamma\right|\right)$$

hold uniformly for suitable functions f with a positive constant C being independent of f?

To answer this question, we need to develop capacities in the Gaussian setting.

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2. Gaussian-Sobolev spaces and capacities

Gaussian-Sobolev spaces

Definition 1 (Gaussian-Sobolev spaces)

Let $p \in [1, \infty]$. Define the Gaussian-Sobolev space $W^{1,p}(\mathbb{G}^n)$ to be the class of all $f \in L^p(\mathbb{G}^n)$ satisfying that $\nabla f \in L^p(\mathbb{G}^n)$. For any $f \in W^{1,p}(\mathbb{G}^n)$, define

$$\|f\|_{W^{1,p}(\mathbb{G}^n)} := \begin{cases} \left(\|f\|^p_{L^p(\mathbb{G}^n)} + \|\nabla f\|^p_{L^p(\mathbb{G}^n)}\right)^{\frac{1}{p}} & \text{as} \quad p \in [1, \infty); \\ \|f\|_{L^\infty(\mathbb{G}^n)} + \|\nabla f\|_{L^\infty(\mathbb{G}^n)} & \text{as} \quad p = \infty. \end{cases}$$

- It is easy to show that for any $f \in C_c^1(\mathbb{R}^n)$ and $p \in [1, \infty)$,

$$\|f\|_{W^{1,p}(\mathbb{G}^n)} \sim \|\nabla f\|_{L^p(\mathbb{G}^n)} + \|f\|_{L^1(\mathbb{G}^n)} \simeq \|\nabla f\|_{L^p(\mathbb{G}^n)} + \left|\int_{\mathbb{R}^n} f \, dV_\gamma\right|.$$

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Density Properties of Sobolev spaces

Density Properties

Let $p \in [1, \infty)$. Then

(i) the set $C_c^1(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{G}^n)$, namely, for any $f \in W^{1,p}(\mathbb{G}^n)$, there exists a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subset C_c^1(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \|f_j - f\|_{W^{1,p}(\mathbb{G}^n)} = 0;$$

(ii) the set

$$\left\{ f \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f \, dV_\gamma = 0 \right\}$$

is dense in

$$\left\{ f \in W^{1,p}(\mathbb{G}^n) : \int_{\mathbb{R}^n} f \, dV_\gamma = 0 \right\}.$$

Gaussian-Sobolev capacity

Definition 2 (Gaussian-Sobolev capacity)

Let $p \in [1, \infty]$ and $E \subset \mathbb{R}^n$ be an arbitrary set. Let

$$\mathcal{A}_p(E) := \left\{ f \in W^{1,p}(\mathbb{G}^n) : \ E \subset \{x \in \mathbb{R}^n : f(x) \geq 1\}^\circ \right\}.$$

Define the Gaussian-Sobolev p-capacity of E as:

$$\mathrm{Cap}_p(E;\, \mathbb{G}^n) := \begin{cases} \inf \left\{ \|f\|^p_{W^{1,p}(\mathbb{G}^n)} : \ f \in \mathcal{A}_p(E) \right\}, & p \in [1, \infty); \\ \inf \left\{ \|f\|_{W^{1,\infty}(\mathbb{G}^n)} : \ f \in \mathcal{A}_\infty(E) \right\}, & p = \infty. \end{cases}$$

$*$ Obviously, $\mathrm{Cap}_p(E) \gtrsim V_\gamma(E)$.

Equivalent descriptions

Let $p \in [1, \infty)$ and $E \subset \mathbb{R}^n$ be an arbitrary set. Then

$$\begin{aligned}\mathrm{Cap}_p(E;\,\mathbb{G}^n) &\sim \inf\left\{\left[\|\nabla f\|_{L^p(\mathbb{G}^n)} + \|f\|_{L^1(\mathbb{G}^n)}\right]^p :\ \ f \in \mathcal{A}_p(E)\right\}\\ &\sim \inf\left\{\left[\|\nabla f\|_{L^p(\mathbb{G}^n)} + \left|\int_{\mathbb{R}^n} f\, dV_\gamma\right|\right]^p :\ \ f \in \mathcal{A}_p(E)\right\}\\ &\sim \inf\left\{\|f\|^p_{W^{1,p}(\mathbb{G}^n)} :\ \ f \geq 0,\ f \in \mathcal{A}_p(E)\right\}.\end{aligned}$$

Basic properties of capacities

Let $p \in [1, \infty)$. Then Cap_p satisfies:

(i) $\mathrm{Cap}_p(\emptyset\,;\ \mathbb{G}^n) = 0$ and $\mathrm{Cap}_p(\mathbb{R}^n\,;\ \mathbb{G}^n) \le 1$.

(ii) If $E_1 \subseteq E_2 \subset \mathbb{R}^n$, then $\mathrm{Cap}_p(E_1;\ \mathbb{G}^n) \le \mathrm{Cap}_p(E_2;\ \mathbb{G}^n)$.

(iii) For any sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of $\mathbb{R}^n$,

$$\mathrm{Cap}_p\left(\bigcup_{j=1}^{\infty} E_j;\ \mathbb{G}^n\right) \le \sum_{j=1}^{\infty} \mathrm{Cap}_p(E_j;\ \mathbb{G}^n).$$

(iv) For any $1 \le p < q < \infty$ and any set $E \subset \mathbb{R}^n$,

$$2^{-1/p}[\mathrm{Cap}_p(E;\ \mathbb{G}^n)]^{1/p} \le 2^{-1/q}[\mathrm{Cap}_q(E;\ \mathbb{G}^n)]^{1/q}.$$

(v) for $p \in (1, \infty)$ and any Suslin set E,

$$\mathrm{Cap}_p(E;\, \mathbb{G}^n) = \sup\{\mathrm{Cap}_p(K;\, \mathbb{G}^n) :\ \text{compact } K \subset E\};$$

(vi) for $p \in (1, \infty)$ and any set E,

$$\mathrm{Cap}_p(E;\, \mathbb{G}^n) = \inf\{\mathrm{Cap}_p(O;\, \mathbb{G}^n) :\ \text{open } O \supset E\}.$$

• Suslin set (analytic set) — a continuous image of a Polish space (separable completely metrizable topological space)

(vii) For any sequence $\{K_j\}_{j=1}^{\infty}$ of compact subsets of $\mathbb{R}^n$ such that $K_1 \supseteq K_2 \supseteq \cdots$,

$$\lim_{j\to\infty} \mathrm{Cap}_p\left(K_j;\ \mathbb{G}^n\right) = \mathrm{Cap}_p\left(\bigcap_{j=1}^{\infty} K_j;\ \mathbb{G}^n\right).$$

(viii) When $p \in (1, \infty)$, for any sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of $\mathbb{R}^n$ such that $E_1 \subseteq E_2 \subseteq \cdots$,

$$\lim_{j\to\infty} \mathrm{Cap}_p\left(E_j;\ \mathbb{G}^n\right) = \mathrm{Cap}_p\left(\bigcup_{j=1}^{\infty} E_j;\ \mathbb{G}^n\right).$$

$*$ (vii) can be used to show the sublinearity of the functional

$$f \mapsto \int_{\mathbb{R}^n} f\, d\mathrm{Cap}_p := \int_0^{\infty} \mathrm{Cap}_p\{x \in \mathbb{R}^n :\ f(x) > \lambda\}\, d\lambda$$

for positive functions f.

∞-capacities

Properties of ∞-capacities

For any set $E \subset \mathbb{R}^n$,

$$\lim_{p\to\infty} [\mathrm{Cap}_p(E;\, \mathbb{G}^n)]^{1/p} \le \mathrm{Cap}_\infty(E;\, \mathbb{G}^n) \le 2 \lim_{p\to\infty} [\mathrm{Cap}_p(E;\, \mathbb{G}^n)]^{1/p}.$$

and

$$\mathrm{Cap}_\infty(E;\, \mathbb{G}^n) \sim \inf\left\{ \|\nabla f\|_{L^\infty(\mathbb{G}^n)} + \|f\|_{L^1(\mathbb{G}^n)} :\ f \in \mathcal{A}_\infty(E) \right\}$$
$$\sim \inf\left\{ \|\nabla f\|_{L^\infty(\mathbb{G}^n)} + \left| \int_{\mathbb{R}^n} f \, dV_\gamma \right| :\ f \in \mathcal{A}_\infty(E) \right\}.$$

3. Capacitary characterization of Sobolev embeddings

Capacitary inequality

A capacitary inequality

Let $1 \le p < \infty$ and $f \in W^{1,p}(\mathbb{G}^n)$ be continuous. For any $t \in (0, \infty)$ set

$$E_t(f) := \{x \in \mathbb{R}^n : |f(x)| > t\}.$$

Then

$$\int_0^\infty \mathrm{Cap}_p(E_t(f); \mathbb{G}^n)\, dt^p \lesssim \|f\|^p_{W^{1,p}(\mathbb{G}^n)}.$$

*

$$W^{1,p}(\mathbb{G}^n) \subset L^p(\mathrm{Cap}_p) \subset L^p(V_\gamma)$$

Theorem 1

Let $1 \leq p \leq q < \infty$ and μ be a non-negative Radon measure. Then the following two assertions are equivalent.

(i) There exists a positive constant C such that for all compact sets $K \subset \mathbb{R}^n$,

$$\mu(K) \leq C[\mathrm{Cap}_p(K; \mathbb{G}^n)]^{q/p}.$$

(ii) There exists a positive constant C such that for all functions $f \in C(\mathbb{R}^n) \cap W^{1,p}(\mathbb{G}^n)$,

$$\left(\int_{\mathbb{R}^n} |f|^q \, d\mu\right)^{1/q} \leq C\|f\|_{W^{1,p}(\mathbb{G}^n)}.$$

Theorem 2

Let $p \in [1, \infty)$, $0 < q < p < \infty$ and μ be a non-negative Radon measure. Then the following two conditions are equivalent:

(i) The capacitary minimizing function

$$h_{\mu,p}(t) := \inf \left\{ \mathrm{Cap}_p(K; \mathbb{G}^n) : K \text{ is compact with } \mu(K) \geq t \right\}$$

satisfies

$$\|h_{\mu,p}\| := \left(\int_0^\infty \frac{dt^{p/(p-q)}}{[h_{\mu,p}(t)]^{q/(p-q)}} \right)^{(p-q)/p} < \infty.$$

(ii) There exists a positive constant C such that for all functions $f \in C(\mathbb{R}^n) \cap W^{1,p}(\mathbb{G}^n)$,

$$\left(\int_{\mathbb{R}^n} |f|^q \, d\mu \right)^{1/q} \leq C \|f\|_{W^{1,p}(\mathbb{G}^n)}.$$

4. Capacitary-isoperimeter and -Poincaré inequalities

Gaussian Minkowski content

For every Borel set $A \subset \mathbb{R}^n$, define the Gaussian Minkowski content of its boundary ∂A as

$$\mathcal{O}_{n-1}(\partial A) := \liminf_{r \to 0} \frac{V_\gamma(A_r) - V_\gamma(A)}{r},$$

with $A_r := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \le r\}$.

$*$ If the Borel set $A \subset \mathbb{R}^n$ has smooth boundary ∂A, then

$$\mathcal{O}_{n-1}(\partial A) = \int_{\partial A} \gamma(x)\, d\mathcal{H}_{n-1}(x).$$

$*$ The Cheeger isoperimetric inequality on $\mathbb{G}^n$: for any Borel set $A \subset \mathbb{R}^n$ with smooth boundary ∂A,

$$\frac{\mathcal{O}_{n-1}(\partial A)}{V_\gamma(A)\, V_\gamma(\mathbb{R}^n \setminus A)} \ge 2\sqrt{\frac{2}{\pi}}.$$

[L96] **M. Ledoux**, Lecture Notes in Math. 1648, Springer, Berlin, 1996, 165-264.

Theorem 3

If $K \subset \mathbb{R}^n$ is compact, then

$$\mathrm{Cap}_1(K;\mathbb{G}^n) = \inf\Big\{\mathcal{O}_{n-1}(\partial O) + V_\gamma(O) : \text{open } O \supset K \text{ with compact } \overline{O} \text{ and smooth } \partial O\Big\}.$$

- Tool: Coarea formula for the Gaussian space [L96]:

For a smooth function f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\nabla f|\, dV_\gamma = \int_0^\infty \left(\int_{\{x\in\mathbb{R}^n:\, |f(x)|=s\}} \gamma(x)\, d\mathcal{H}_{n-1}(x) \right) ds,$$

where $d\mathcal{H}_{n-1}$ is the Hausdorff measure of dimension $n-1$ on the surface $\{x \in \mathbb{R}^n : \ |f(x)| = s\}$.

Theorem 4

The following three statements are equivalent:

(i) (Cheeger isoperimetric inequality) For any open set $O \subset \mathbb{R}^n$ with smooth boundary,

$$\mathcal{O}_{n-1}(\partial O) \geq 2\sqrt{\frac{2}{\pi}} V_\gamma(O) V_\gamma(\mathbb{R}^n \setminus O).$$

(ii) For any smooth function f,

$$\int_{\mathbb{R}^n} |\nabla f| \, dV_\gamma \geq 2\sqrt{\tfrac{2}{\pi}} \int_{-\infty}^{\infty} V_\gamma(\{x \in \mathbb{R}^n : \ f(x) > s\}) \times V_\gamma(\{x \in \mathbb{R}^n : \ f(x) \leq s\}) \, ds.$$

(iii) (Gaussian 1-Poincaré/Sobolev inequality) For any smooth function f with $\int_{\mathbb{R}^n} f \, dV_\gamma = 0$,

$$\int_{\mathbb{R}^n} |\nabla f| \, dV_\gamma \geq \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^n} |f| \, dV_\gamma.$$

- Key tool:

$*$ Doubling case

— The boxing inequality (unknown for the Gaussian setting);

$*$ Gaussian case

— Capacities $\mathrm{Cap}_{\mathrm{BV}}$ of functions of bounded variations;

— Equivalence $\mathrm{Cap}_1 \sim \mathrm{Cap}_{\mathrm{BV}}$

Thank you for your attention!