Explicit regularity estimates for solutions to quasi-linear PDEs

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Model problem: p-Poisson equation
p-Laplacian
Variational formulation
Approximation spaces
Local regularity
Local Hölder regularity
Locally weighted Hölder regularity
Local L_p-Lipschitz regularity
Global regularity
What can be expected?
Global Besov regularity
Further improvements $(1 < p \leq 2)$
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Model problem: The p-Poisson equation

We consider weak solutions $u \in W^1_{\mathrm{loc}}(L_p(\Omega))$ to

$$\mathcal{A}_p(u) = f \qquad in \qquad \Omega$$

with

- $1 < p < \infty$,
- $\mathcal{A}_p(u)$ ... p-Laplacian,
- $f \in L_{q,\mathrm{loc}}(\Omega)$, where $q \geq p'$ and $1/p + 1/p' = 1$,
- $\Omega \subseteq \mathbb{R}^d$ ... domain (connected open set).

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p-Laplacian $\mathcal{A}_p$ for $1 < p < \infty$

$$u \mapsto \mathcal{A}_p(u) = \mathrm{div}\left(|\nabla u|_2^{p-2}\,\nabla u\right)$$

- **quasi**-linear, elliptic, 2nd order partial differential op.
- model character like the Laplacian for linear PDEs
- $p = 2 \quad \Longrightarrow \quad \mathcal{A}_p(u) = \mathrm{div}(\nabla u) = \Delta u$
- applications in models, e.g., for
  - turbulent flows of a gas in porous media
  - radiation of heat
  - non-Newtonian fluid theory
  - ...

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The corresponding variational formulation

$$\int_\Omega \left\langle |\nabla u|_2^{p-2} \nabla u, \nabla \varphi \right\rangle_{\mathbb{R}^d} \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega)$$

admits a unique weak solution $u \in W^1(L_p(\Omega))$ if

- $\Omega \subset \mathbb{R}^d$ is a bounded domain,
- $f \in W^{-1}(L_{p'}(\Omega))$,
- (sufficiently regular) boundary values are prescribed.

(see Lions 1969)

$\Longrightarrow$ Meaningful to ask for (maximal) regularity of u with

$$\mathcal{A}_p(u) = f \in L_q(\Omega) \qquad in \qquad \Omega$$

for bounded Lipschitz domains Ω and $q \geq p'$ in scales of Sobolev or Besov spaces.

Uniform vs. adaptive approximation

$$A^{s/d}_{\mathrm{uniform}}(L_p(\Omega)) \approx W^s(L_p(\Omega)), \qquad s > 0$$

$$A^{\sigma/d}_{\mathrm{adaptive}}(L_p(\Omega)) \approx B^{\sigma}_{\tau_\sigma}(L_{\tau_\sigma}(\Omega)), \qquad \sigma > 0, \quad \frac{1}{\tau_\sigma} = \frac{\sigma}{d} + \frac{1}{p}$$

(vaild for FEM and wavelet methods!)

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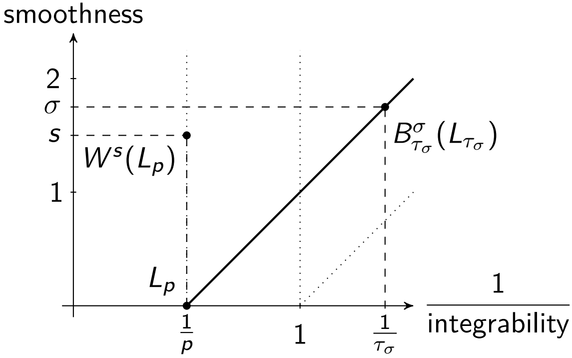

$\Longrightarrow$ Adaptivity pays off iff $\sigma > s$

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Local Hölder regularity

Let

- $1 < p < \infty$,
- $\Omega \subset \mathbb{R}^d$ ... bounded domain ($d \geq 2$),
- $f \in L_q(\Omega)$ with $q > d$.

Then every

$$u \in W^1(L_p(\Omega)) \quad \text{with} \quad \mathcal{A}_p(u) = f \quad \text{in} \quad \Omega$$

satisfies

$$u \in C^{1,\alpha}_{\text{loc}}(\Omega) \quad \text{for some} \quad \alpha \in (0,1).$$

(see, e.g., Diening/Kaplický/Schwarzacher 2012)

In $d = 2$ also the *maximal* such $\alpha =: \overline{\alpha} := \overline{\alpha}_{p,q}$ is known (see, e.g., Kuusi/Mingione 2014, Lindgren/Lindqvist 2006).

Locally weighted Hölder regularity

Let additionally

- $\Omega \subset \mathbb{R}^2$ ... bounded Lipschitz domain,
- $s > \max\{1, 2/p\}$.

Then every

$$u \in W^s(L_p(\Omega)) \qquad \text{with} \qquad \mathcal{A}_p(u) = f \quad \text{in} \quad \Omega$$

satisfies expl. bounds on Hölder semi-norms on balls $B \subset\subset \Omega$
$\Longrightarrow$ membership in locally *weighted* Hölder spaces $C^{1,\overline{\alpha}}_{\mathrm{loc},\gamma}(\Omega)$:

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$\Longrightarrow$ membership in locally *weighted* Hölder spaces $C^{1,\overline{\alpha}}_{\mathrm{loc},\gamma}(\Omega)$:

$$\sup_{h \in \mathbb{R}^d} \frac{1}{|h|_2^{\overline{\alpha}}} \left\| \left| \Delta_h \nabla u \right|_2 \,\middle|\, L_\infty(B^h) \right\| = \sup_{\substack{x,y \in B \\ x \neq y}} \frac{|\nabla u(x) - \nabla u(y)|_2}{|x-y|_2^{\overline{\alpha}}} \lesssim c \left(\|f \mid L_q(\Omega)\|^{1/(p-1)} + \left\| |\nabla u|_2 \,\middle|\, W^{s-1}(L_p(\Omega)) \right\| \right),$$

where $c := \mathrm{dist}(B, \partial\Omega)^{-\gamma}$ with (known) $\gamma = \gamma(s, p, \overline{\alpha}) > 0$.
(see Dahlke/Diening/Hartmann/Scharf/W. 2016)

Theorem (Local L_p-Lipschitz regularity)

For $d \in \mathbb{N}$ let $\Omega \subseteq \mathbb{R}^d$ be any domain, $1 < p < \infty$, $f \in L_{p',\mathrm{loc}}(\Omega)$, and $u \in W^1_{\mathrm{loc}}(L_p(\Omega))$ with

$$\mathcal{A}_p(u) = f \qquad \text{in} \qquad \Omega.$$

If $\omega \subsetneq \mathcal{O} \subseteq \Omega$ with $\varepsilon := \mathrm{dist}(\omega, \partial\mathcal{O}) \in (0,\infty]$, $f \in L_{p'}(\mathcal{O})$, and $u \in W^1(L_p(\mathcal{O}))$, then

$$\sup_{|h|_2 \leq \min\{\varepsilon,1\}} \frac{1}{|h|_2^{\beta_p}} \left\| |\Delta_h \nabla u|_2 \mid L_p(\omega) \right\| \lesssim_{d,p} C,$$

with $\quad \beta_p := 1$ *if* $p \leq 2$, $\quad \beta_p := 1/(p-1)$ *for* $p > 2$, $\quad$ *and*

$$C := \left\| |\nabla u|_2 \mid L_p(\mathcal{O}) \right\| \left[\left[\frac{\| f \mid L_{p'}(\mathcal{O}) \|}{\left\| |\nabla u|_2 \mid L_p(\mathcal{O}) \right\|^{p-1}} \right]^{\beta_p} + \max\{1, \varepsilon^{-\beta_p}\} \right].$$

Proof: Localization to cubes & "Nirenberg's translation method"
(Refinement of results due to De Thelin 1982, Pucci/Servadei 2008)

$\Longrightarrow$ **explicit bound** on the growth as $\varepsilon = \mathrm{dist}(\omega, \partial\mathcal{O}) \to 0$

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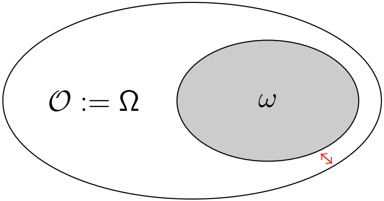
$$\beta_p = \begin{cases} 1 & \text{if} \quad 1 < p \leq 2, \\ \dfrac{1}{p-1} & \text{if} \quad 2 < p < \infty. \end{cases}$$

Corollary (Local regularity $1 + \beta_p$ in L_p)

$$|u|_{B^{1+\beta_p}_\infty(L_p(\omega))} \lesssim_{d,p} C < \infty$$

with $C = C(\mathrm{dist}(\omega, \partial\mathcal{O})^{-\beta_p}, p, \nabla u, f)$ *as above.*



Global regularity - What can be expected?

For $d \in \mathbb{N}$ let $\Omega \subseteq \mathbb{R}^d$ be any Lipschitz domain or $\mathbb{R}^d$ itself.

Theorem (Limited regularity for $p > 2$)

For all $p > 2$ and $\delta > 0$ there exists $u \in W_0^1(L_p(\Omega))$ s.t.

$$\mathcal{A}_p(u) =: f \in L_\infty(\Omega) \quad \text{has comp. supp.} \quad (\Longrightarrow f \in L_{p'}(\Omega))$$

and for all $0 < t, q \leq \infty$

$$u \in B_\infty^{1+\beta_p}(L_t(\Omega)) \setminus B_q^{1+\beta_p+\delta}(L_t(\Omega))$$

(provided that $1 + \beta_p > d \max\{0, 1/t - 1\}$).

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(provided that $1 + \beta_p > d \max\{0, 1/t - 1\}$).

$\Longrightarrow$ Independent of the integrability, smoothness $1 + \beta_p$ cannot be improved in general (even for smooth domains)!

(In particular: $u \notin C_{\mathrm{loc}}^{1,\beta_p+\delta}(\Omega)$, i.e. $\overline{\alpha} \leq \beta_p$)

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Theorem (Global Besov regularity)

For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $1 < p < \infty$, and $1 \leq s < 1 + \beta_p$.

Then there exists $s < \overline{\sigma} \leq 1 + \beta_p$ *such that every*

$$u \in W^s(L_p(\Omega)) \quad \text{with} \quad \mathcal{A}_p(u) = f \in L_{p'}(\Omega) \quad \text{in} \quad \Omega,$$

satisfies $u \in B^{\sigma}_{\tau_\sigma}(L_{\tau_\sigma}(\Omega))$ *for all*

$$0 < \sigma < \overline{\sigma}, \qquad \frac{1}{\tau_\sigma} = \frac{\sigma}{d} + \frac{1}{p}.$$

(incl. suitable quasi-norm estimates)

In detail:

$$\overline{\sigma} = \overline{\sigma}(s, p, d) := \max_{c \in [0,1]} \min\left\{ s\frac{d}{d-c},\, 1 + \beta_p(1-c) \right\}$$

Further improvements for $1 < p \leq 2$

Theorem (Kondratiev regularity)

For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $1 < p \leq 2$ and $f \in L_{p'}(\Omega)$. Then the unique solution to

$$\begin{aligned} \mathcal{A}_p(u) &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

satisfies $u \in \mathcal{K}^2_{1,p}(\Omega)$.

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satisfies $u \in \mathcal{K}^2_{1,p}(\Omega)$.

Corollary

If additionally $d = 2$, then

$$u \in B^{\sigma}_{\tau_\sigma}(L_{\tau_\sigma}(\Omega))$$

for all

$$0 < \sigma < \overline{\sigma} := \overbrace{2}^{=1+\beta_p}, \qquad \frac{1}{\tau_\sigma} = \frac{\sigma}{2} + \frac{1}{p}.$$

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Previously (in Dahlke/Diening/Hartmann/Scharf/W. 2016), we were able to show that **adaptivity pays off**, i.e.,

$$\overline{\sigma} > s, \tag{1}$$

provided that

$$u \in W^s(L_p(\Omega)) \cap C^{1,\overline{\alpha}}_{\mathrm{loc},\gamma}(\Omega) \tag{2}$$

with large enough s and sufficiently small γ.
Drawback: (2) has been verified only for polygonal domains in $d = 2$ and $f \in L_q$ with $q \gg p'$.

Now, we have (1) for every Lipschitz domain ($d \in \mathbb{N}$), all reasonable s and $q = p'$.
Drawback: Expression $\overline{\sigma} = \overline{\sigma}(s, p, d)$ looks unnatural

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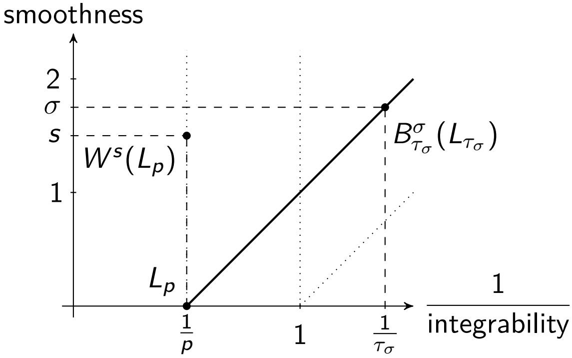
Summary

In this talk, we ...

- ... presented local Hölder/L_p-Lipschitz regularity results for solutions to the p-Poisson equation,
- ... deduced global regularity of these solutions in the adaptivity scale of Besov spaces,
- ... showed that adaptivity always pays off for these nonlinear problems.



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Thank you!