Structured Banach frame decompositions of decomposition spaces

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New perspectives in the theory of function spaces and their applications

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Applied Functional Analysis Group
Technische Universität Berlin

Outline

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1. Motivation (4 slides)
2. Structured Banach frame decompositions of decomposition spaces (4 slides)
3. Example applications (4 slides)

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Applications: Sequence spaces are often friendlier than function spaces.
$\rightsquigarrow$ Use Banach frames and atomic decompositions to study e.g.

- Embeddings,
- Boundedness of operators,
- Entropy numbers,
- Interpolation spaces.

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- **Shearlets** are BFDs for **Shearlet smoothness spaces**. (New result!)
- BFDs for α**-modulation spaces** are also known.

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- Our approach:
  - single (finitely) generated systems,
  - (possibly) compactly supported elements,
  - no continuous frame needed,
  - very general, but assumptions sometimes painful to check (cf. coorbit theory).

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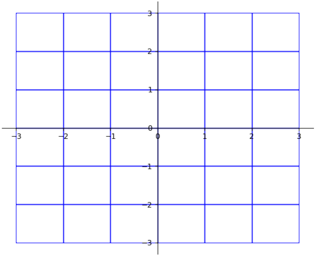
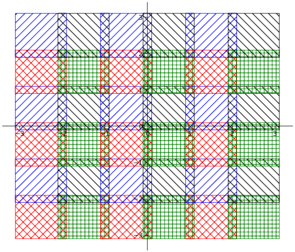
Example: Covering $\mathcal{Q}$ for **Modulation spaces**:



or



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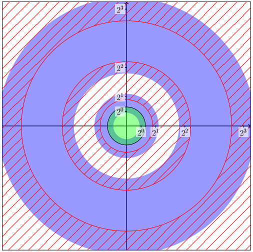

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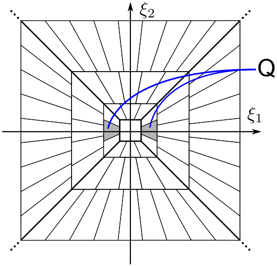
Example: Covering for **shearlet smoothness spaces** (image courtesy of Martin Genzel)



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Examples: Gabor systems, wavelet systems, shearlet systems, etc.

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Note: On the Fourier side, we have

$$\mathcal{F} \gamma^{[i]} = |\det T_i|^{-1/2} \cdot L_{b_i} \left(\widehat{\gamma} \circ T_i^{-1} \right).$$

Hence, if $\widehat{\gamma}$ is concentrated in $Q \subset \mathbb{R}^d$, then $\mathcal{F} \gamma^{[i,k;\delta]}$ is concentrated in $T_i Q + b_i = Q_i$.

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Goal: Find sequence space $C_w^{p,q}$ and **conditions on** γ (compatible with compact support) such that the structured system $\Gamma^{(\delta)}$ is a BFD for the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$.

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The **decomposition space** determined by $\mathcal{Q}, p, q, w$ is

$$\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) := \left\{ g \in \mathcal{R} \,\middle|\, \|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_w^q)} < \infty \right\}.$$

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Hope: If γ is nice, then $\Gamma^{(\delta)} = \left(\gamma^{[i,k;\delta]}\right)_{i\in I, k\in\mathbb{Z}^d}$ is a BFD for $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, where

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Structured Banach frame decompositions: General questions

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Note: Even for characterizing L^2, the required "niceness" depends heavily on $\mathcal{Q}$:

- For Gabor systems: Sufficient if γ belongs to the Wiener space.
- For wavelets, γ has to have vanishing moments.

Structured Banach frames — The theorem

Theorem (FV; 2016)

Let $w = (w_i)_{i \in I}$ *be* $\mathcal{Q}$*-moderate and let* $p, q \in (0, \infty]$.

If

then there is $\delta_0 > 0$*, such that for* $0 < \delta \leq \delta_0$*, the family* $(L_{\delta \cdot T_i^{-T} k}\, \widetilde{\gamma^{[i]}})_{i,k}$*, with* $\tilde{g}(x) = g(-x)$*, forms a* Banach frame *for* $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$*, with coeff. space* $C_w^{p,q}$.

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with

$$M_{j,i} \sim \qquad \cdot \left(\fint_{Q_i} \left| [\quad \widehat{\gamma}] \left(T_j^{-1} (\xi - b_j) \right) \right| \mathrm{d}\xi \right)^{\tau},$$

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$$M_{j,i} := \left(\frac{w_j}{w_i}\right)^{\tau} \cdot (1 + \|T_j^{-1} T_i\|)^{\sigma} \cdot \left(\fint_{Q_i} \max_{\substack{|\alpha| \leq N \\ |\beta| \leq 1}} \left| [\partial^\alpha \widehat{\partial^\beta \gamma}] \left(T_j^{-1}(\xi - b_j)\right) \right| d\xi \right)^{\tau},$$

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Structured atomic decompositions

Theorem (FV; 2016)

Under similar conditions on γ as before, there is some $\delta_0 > 0$ such that the family $\Gamma^{(\delta)} = \left(\gamma^{[i,k;\delta]}\right)_{i,k}$ is an atomic decomposition of $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, with coefficient space $C_w^{p,q}$, if $\delta \in (0, \delta_0]$.

Outline

New proofs of classical results

Corollary

*For **Besov spaces** $B_s^{p,q}(\mathbb{R}^d)$:*

- *"Horrible condition" $\rightsquigarrow$ smoothness + localization + **vanishing moments**.*
- *Structured system $\rightsquigarrow$ Wavelet system.*

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- *Precisely: Suffices to have*

$$|\partial^\alpha \widehat{\gamma}(\xi)| \lesssim (1+|\xi|)^{-L_1} \cdot \min\left\{1, |\xi|^{L_2}\right\} \qquad \text{for } |\alpha| \leq \left\lceil \frac{d+\varepsilon}{\min\{1,p\}} \right\rceil,$$

with

  - $L_2 > s$ *to get Banach frames,*
  - $L_2 > \left(p^{-1}-1\right)_+ \cdot d - s$ *to get atomic decompositions.*

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Theorem (FV; 2016)

For α***-modulation spaces*** $M_{s,\alpha}^{p,q}(\mathbb{R}^d)$*:*

- *"Horrible condition"* $\rightsquigarrow$ *smoothness + localization.*
- *Structured system: Nice structured system (for* $\alpha = 0$*:* ***Gabor system****).*

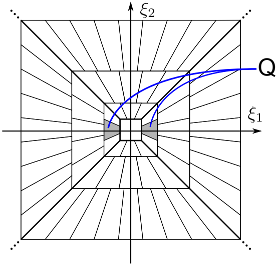
Shearlet smoothness spaces

Shearlet smoothness spaces

This is joint work with **Anne Pein**

Shearlet smoothness spaces

The **shearlet covering** $\mathcal{S} = (T_i Q + b_i)_{i \in I}$:



The spaces $\mathscr{S}_s^{p,q}(\mathbb{R}^2) = \mathcal{D}(\mathcal{S}, L^p, \ell_{w^s}^q)$ are called **shearlet smoothness spaces** **(Labate, Mantovani, Negi; 2013).**

The weight w is chosen such that $w_i \sim 1 + |\xi|$ for $\xi \in Q_i$.

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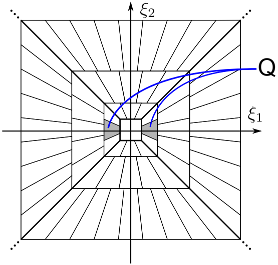

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Observation: Structured family generated by ψ is a **cone-adapted shearlet system** (Guo, Labate, Kutyniok; 2006):

$$\Psi^{(\delta)} := \left(\psi^{[i,k,\delta]}\right)_{i \in I, k \in \mathbb{Z}^2} = \mathrm{SH}\left(M_{b_0}(\psi \circ T_0^T), \psi; \delta\right).$$

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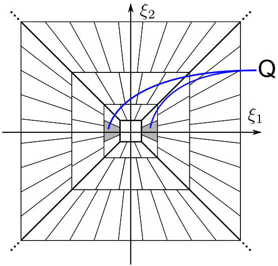

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Theorem (Pein, FV; 2017)

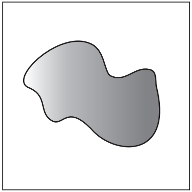
Let $p_0, q_0 \in (0,1]$ and $s_0 \geq 0$. There are $N_1, N_2 \in \mathbb{N}$ such that if $\psi_1, \psi_2 \in C_c^{N_1}(\mathbb{R})$ and $\psi = \psi_1 \otimes \psi_2$ with

$$\widehat{\psi}(\xi) \neq 0 \text{ for } \xi \in \overline{Q} \quad \text{and} \quad \left.\frac{\mathrm{d}^\ell}{\mathrm{d}\,\xi^\ell}\right|_{\xi=0} \widehat{\psi_1}(\xi) = 0 \text{ for } \ell = 0, \ldots, N_2$$

*then there is $\delta_0 > 0$, such that $\Psi^{(\delta)}$ is a Banach frame **and** an atomic decomposition for $\mathscr{S}_s^{p,q}(\mathbb{R}^2)$ for all $p \geq p_0$, $q \geq q_0$, $|s| \leq s_0$, and $0 < \delta \leq \delta_0$.*

Application: Approximation of cartoon-like functions

We consider C^2 **cartoon-like functions** $f \in \mathcal{E}^2$:



(image courtesy of Gitta Kutyniok)

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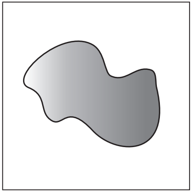

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- the **analysis coefficients** of f w.r.t. $\mathrm{SH}(\psi;\delta)$ are **ℓ^p-summable** for $p > \frac{2}{3}$.
- For suitable linear combination f_N of N elements of the **dual shearlet frame**:

$$\|f - f_N\|_{L^2} \leq C_{\delta,\psi} \cdot N^{-1} \cdot (1 + \ln N)^{3/2} .$$

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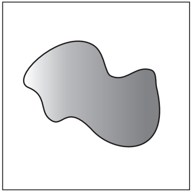

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New result (Pein, FV; 2017)

For suitable linear comb. g_N of N elements **of the shearlet frame** $\mathrm{SH}(\psi;\delta)$:

$$\|f - g_N\|_{L^2} \leq C_{\varepsilon,\delta,\psi} \cdot N^{-(1-\varepsilon)} \qquad \forall \varepsilon \in (0,1) \text{ and } N \in \mathbb{N}.$$

Conclusion

In this talk:

- We presented a framework for constructing **structured, singly generated, (possibly) compactly supported Banach frame decompositions** for the Besov-type decomposition spaces $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, where $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$.

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- Apply general framework to other cases (e.g.: anisotropic Besov spaces, spaces related to wave packets, ...).
- Extend the framework to the case of Triebel-Lizorkin type spaces.

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Questions, comments, counterexamples?

☺