Fourier multipliers and stability of semigroups

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Based on joint work with Jan Rozendaal (Warsaw/Canberra):

Introduction

In Lebeau 1996 considered a damped wave equation on a manifold M:

$$\begin{cases} u_{tt}(t,x) + a(x)u_t(t,x) - \Delta u(t,x) = 0, & x \in M \\ u = 0, & x \in \partial M \\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1 & x \in M. \end{cases}$$

M - smooth compact connected Riemannian manifold with boundary,
Damping: $a(x) \geqslant 0$, $0 < \|a\|_\infty < \infty$.
Reformulation as Cauchy problem.

$$X = W^{1,2}_0(M) \times L^2(M), \quad D(A) = (W^{2,2}(M) \cap W^{1,2}_0(M)) \cap W^{1,2}_0(M).$$

$$U_t + AU = 0, \quad U(0) = (u_0, u_1), \quad A = \begin{pmatrix} 0 & -I \\ -\Delta & a \end{pmatrix}$$

By Hille–Yosida: $-A$ generates a C_0-semigroup: $\|S(t)\| \leqslant 1$.
Lebeau 1996 showed: $\|e^{-tA}A^{-1}\| \leqslant \frac{C\log(3+\log(3+t))}{\log(t+3)}$

This means classical solutions have log decay!

Last 20 years: detailed study of decay of classical solutions.

Theorem (Batty and Duyckaerts 2008)

Let $S(t) = e^{-tA}$ be a bounded C_0-semigroup on a Banach space X, $\sigma(A) \subseteq \mathbb{C}_+$, and

$$M(s) = \sup_{|\xi|\leqslant s} \|(i\xi + A)^{-1}\|, \quad \widetilde{M}(s) = M(s)\big[\log(1 + M(s)) + \log(1 + s)\big].$$

Then there exist $k, K, c, C > 0$ such that

$$\frac{c}{M^{-1}(kt)} \leqslant \|S(t)A^{-1}\| \leqslant \frac{C}{\widetilde{M}^{-1}(Kt)}.$$

Characterizations without log-terms if X is a Hilbert space:

- Borichev–Tomilov 2010 $M(s) \eqsim s^{\beta}$
- Batty–Chill-Tomilov 2014 $M(s) \eqsim s^{\beta} \times$ log variations
- Rozendaal–Seiffert–Stahn 2017 $\exists \lambda > 1$ s.t. $\liminf_{s\to\infty} \frac{M(\lambda s)}{M(s)} > 1$.

A simple case of the converse implication

Decay of $\|e^{-tA}A^{-1}\| \longleftrightarrow$ growth of $\|(is+A)^{-1}\|$.
How to make the implication "$\longrightarrow$" precise in a special case:
Assume $\|e^{-tA}A^{-1}\| \leqslant Ct^{-1-\varepsilon}$ with $\varepsilon > 0$. Then for $s \in \mathbb{R}$,

$$(is+A)^{-1}A^{-1} = \int_0^\infty e^{-its}e^{-tA}A^{-1}\,dt.$$

Therefore,

$$\|(is+A)^{-1}A^{-1}\| \leqslant \int_0^\infty \|e^{-its}e^{-tA}A^{-1}\|dt \leqslant C.$$

Thus $(is+A)^{-1}A^{-1}$ is uniformly bounded. Now write:

$$(is+A)^{-1}A^{-1} = \frac{A^{-1}-(is+A)^{-1}}{is}.$$

This yields $\|(is+A)^{-1}\| \leqslant \widetilde{C}(1+|s|)$.

Overview

Goal: Develop a variation of this theory in the case $\sup_{t \geqslant 0} \|S(t)\| = \infty$.

One known result: Bátkai-Engel-Prüss-Schnaubelt 2006

Examples: operator matrices, multiplication operators on $W^{k,p}$.
Main technique: (L^p, L^q) Fourier multipliers with $m(\xi) = (i\xi + A)^{-k}$.

Fourier multipliers: scalar valued

Given $m : \mathbb{R} \to \mathbb{C}$ locally integrable and of polynomial growth
$\mathscr{S}$ - Schwartz functions, $\mathscr{S}'$ - tempered distributions

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{T_m} & \mathscr{S}' \\ \mathcal{F} \downarrow & & \uparrow \mathcal{F}^{-1} \\ \mathscr{S} & \xrightarrow{m} & \mathscr{S}' \end{array} \qquad (1)$$

We say $m \in \mathcal{M}_{p,q}$ if $T_m \in \mathcal{L}(L^p, L^q)$, $\|m\|_{\mathcal{M}_{p,q}} = \|T_m\|_{\mathcal{L}(L^p,L^q)}$.

- $\mathcal{M}_{2,2} = L^\infty$
- $\mathcal{M}_{p,p} \hookrightarrow L^\infty$ (converse false if $p \neq 2$)
- If $p > q$, then $\mathcal{M}_{p,q} = \{0\}$.
- If $1 < p \leqslant 2 \leqslant q < \infty$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then $L^r \hookrightarrow L^{r,\infty} \hookrightarrow \mathcal{M}_{p,q}$

The last two results are due to Hörmander 1960

Fourier multipliers: vector-valued

Given $m : \mathbb{R}\setminus\{0\} \to \mathcal{L}(X, Y)$ strongly measurable + growth conditions

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}; X) & \xrightarrow{T_m} & \mathscr{S}'(\mathbb{R}; Y) \\ \mathcal{F}\downarrow & & \uparrow \mathcal{F}^{-1} \\ \mathscr{S}(\mathbb{R}; X) & \xrightarrow{m} & \mathscr{S}'(\mathbb{R}; Y) \end{array} \tag{2}$$

We say $m \in \mathcal{M}_{p,q}(\mathbb{R}; X, Y)$ if $T_m \in \mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; Y))$,

$$\|m\|_{\mathcal{M}_{p,q}(\mathbb{R};X,Y)} = \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X),L^q(\mathbb{R};Y))}.$$

Sufficient conditions for $m \in \mathcal{M}_{p,p}$ in vector-valued case:

- McConnell 1984, Bourgain 1984: Mihlin type results if X is UMD
- Weis 2001, Mihlin for operator-valued multipliers + appl. PDE
  Uses R-boundedness of $\{m(\xi) : \xi \in \mathbb{R}\}$ (nec.) and $\{\xi m'(\xi) : \xi \in \mathbb{R}\}$

$\mathcal{M}_{p,q}$ with $p < q$ is very interesting in the operator-valued case and almost nothing is known about them.

Vector-valued Bessel potential spaces $H^{s,p}$

Theorem

(i) *If X is UMD, then $W^{k,p}(\mathbb{R};X) = H^{k,p}(\mathbb{R};X)$*

(ii) *If $W^{k,p}(\mathbb{R};X) = H^{k,p}(\mathbb{R};X)$ with $d \geqslant 2$ or k is odd, then X is UMD*

(i): follows from the vector-valued Mihlin multiplier theorem

(ii): Bourgain 1983 and Geiss–Montgomery-Smith–Saksman [2010]

Theorem (Han–Meyer, 1996)

$F^s_{p,2}(\mathbb{R}^d;X) = H^{s,p}(\mathbb{R}^d;X) \Longleftrightarrow$ *X is isomorphic to a Hilbert space.*

Theorem (Meyries–Veraar, 2015)

If X is UMD, $p \in (1,\infty)$, $s \in (-1/p', 1/p)$, then
$\mathbf{1}_{\mathbb{R}^d_+} : H^{s,p}(\mathbb{R}^d;X) \to H^{s,p}(\mathbb{R}^d;X)$

Weighted setting allowed. Problem: Necessity of UMD ?

Multipliers for spaces with Fourier type

X has Fourier type p if $\mathcal{F} : L^p(\mathbb{R}; X) \to L^{p'}(\mathbb{R}; X)$ (Peetre 1969).
Connection Hausdorff–Young inequalities. Only $p \in [1, 2]$.

Original motivation: comparison of real and complex interpolation:

$$(X_0, X_1)_{\theta,p} \hookrightarrow [X_0, X_1]_\theta \hookrightarrow (X_0, X_1)_{\theta,p'}$$

- Every X has Fourier type 1
- X Fourier type 2 $\Longleftrightarrow$ X is a Hilbert space (Kwapień 1972)
- Fourier type p implies Fourier type u for all $u \in [1, p]$
- $L^s(\Omega)$ with $s \in [1, \infty)$ has Fourier type $\min\{s, s'\}$

Convention: X has Fourier cotype $q \in [2, \infty]$ if X has Fourier type q'.

Multipliers for Banach spaces with Fourier type

Proposition (RV, 2017a)

Assume X has Fourier type $p \in [1,2]$ and Y has Fourier cotype $q \in [2,\infty]$ and set $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then $L^r(\mathbb{R}; \mathcal{L}(X,Y)) \subseteq \mathcal{M}_{p,q}(\mathbb{R}, X, Y)$.

Proof:

$$\|T_m(f)\|_{L^q(\mathbb{R};Y)} \lesssim_{Y,q} \|m\widehat{f}\|_{L^{q'}(\mathbb{R};Y)} \overset{\text{Hölder}}{\leqslant} \|m(\cdot)\|_{L^r(\mathbb{R};\mathcal{L}(X,Y))}\|\widehat{f}\|_{L^{p'}(\mathbb{R};X)}$$
$$\lesssim_{X,p} \|m(\cdot)\|_{L^r(\mathbb{R};\mathcal{L}(X,Y))}\|f\|_{L^p(\mathbb{R};X)}.$$

By interpolation techniques we obtain a result of Hörmander type:

Theorem (RV, 2017a)

Assume X has Fourier type $> p$ and Y has Fourier cotype $< q$ and set $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then $L^{r,\infty}(\mathbb{R}; \mathcal{L}(X,Y)) \subseteq \mathcal{M}_{p,q}(\mathbb{R}, X, Y)$.

Converses: Fourier type is necessary

Problem: limiting case of Theorem with Fourier type p and cotype q

Multipliers for Banach spaces with type and cotype

Rademacher (co)type less restrictive than Fourier (co)type:

- $L^s(\Omega)$ with $s \in [1, \infty)$ has type $\min\{s, 2\}$ and cotype $\max\{s, 2\}$.

Theorem (RV, 2017a)

Assume X has type $> p \in [1, 2)$ and Y has cotype $< q \in (2, \infty]$ and set $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m : \mathbb{R}\backslash\{0\} \to \mathcal{L}(X, Y)$ is strongly measurable and

$$\{|\xi|^{\frac{1}{r}} m(\xi) : \xi \in \mathbb{R}\backslash\{0\}\} \subseteq \mathcal{L}(X, Y)$$

is R-bounded, then $m \in \mathcal{M}_{p,q}(\mathbb{R}, X, Y)$.

Converses: type and cotype necessary
Problem: Limiting case of Theorem?
Yes if X is p-convex (and finite cotype) and Y is q-concave

Main ingredient in the proof

$\gamma(\mathbb{R}; X)$ - completion of the simple functions in the norm

$$\Big\| \sum_{n=1}^N \mathbf{1}_{A_n} x_n \Big\|_{\gamma(\mathbb{R};X)} = \Big\| \sum_{n=1}^N \gamma_n |A_n|^{1/2} x_n \Big\|_{L^2(\Omega;X)}, \quad \gamma_n \sim \mathcal{N}(0,1) \text{ independent.}$$

Theorem (Kalton-Neerven-V.-Weis 2008)

Let X be a Banach space, $p \in [1,2]$ and $s = \frac{1}{p} - \frac{1}{2}$. Then $\dot{B}^s_{p,p}(\mathbb{R}; X) \hookrightarrow \gamma(\mathbb{R}; X) \iff X$ has type p.

Problem: Result with $\dot{B}^s_{p,p}$ replaced by $\dot{H}^{\alpha,r}$?

Result with $\dot{B}^s_{p,p}$ replaced by $\dot{F}^s_{p,\infty}$ is false in general

However, if X has type $p_0 > p$, then with $s_0 = \frac{1}{p_0} - \frac{1}{2}$

$$\dot{H}^{s,p}(\mathbb{R}; X) \hookrightarrow \dot{F}^s_{p,\infty}(\mathbb{R}; X) \hookrightarrow \dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}; X) \hookrightarrow \gamma(\mathbb{R}; X).$$

Main characterization of polynomial stability

From now on $-A$ generates a C_0-semigroup $S(t)$ and $\sigma(A) \subseteq \mathbb{C}_+$.

Theorem (RV, 2017b)

Let $\theta \geqslant 0$ and $n \in \mathbb{N}_0$. The following are equivalent:

1. $\sup_{t\geqslant 0} t^n \|S(t)A^{-\theta}\|_{\mathcal{L}(X)} < \infty$;
2. *There exist $1 \leqslant p \leqslant q \leqslant \infty$ such that for all $k \in \{1, \ldots, n+1\}$*

$$\xi \mapsto (i\xi + A)^k A^{-\theta} \in \mathcal{M}_{p,q}(\mathbb{R}; X)$$

Interpolation argument gives non-integer decay rates.

Polynomial stability results

We say that A has resolvent growth $\beta \geqslant 0$ if $\exists C \geqslant 0$ s.t.

$$\|(\lambda + A)^{-1}\| \leqslant C(|\lambda| + 1)^{\beta}, \quad \lambda \in \mathbb{C}_+.$$

Theorem (RV, 2017b)

Let X be a Hilbert space. Assume has A has resolvent growth $\beta \geqslant 0$. Then for each $\tau \geqslant \beta$ and $\rho = \frac{\tau}{\beta} - 1$, there exists a $C \geqslant 0$ such that

$$\|S(t)A^{-\tau}\| \leqslant Ct^{-\rho}, \quad t \geqslant 1 \quad (*)$$

Bounded semigroups in Hilbert spaces: $\|S(t)A^{-\tau}\| \leqslant Ct^{-\tau/\beta}$.

Theorem (RV, 2017b)

Let X have Fourier type $p \in [1, 2)$ and let $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$. Assume has A has resolvent growth $\beta \geqslant 0$. Then for each $\tau > \beta + r^{-1}$ and $\rho \in [0, \frac{\tau - r^{-1}}{\beta} - 1)$ there exists a $C \geqslant 0$ such that $(*)$ holds.

- Both results follows from the main characterization and the fact that resolvent growth β implies that for all $\tau \geqslant \beta$,

$$\|(\lambda + A)^{-1}A^{-\tau}\| \leqslant \frac{C}{|\lambda|^{\tau-\beta} + 1}, \quad \lambda \in \mathbb{C}_+.$$

  This implies $\xi \mapsto (i\xi + A)^k A^{-\tau} \in \mathcal{M}_{p,p'}(\mathbb{R}; X)$ for suitable τ by the multiplier theorems.
- By interpolation one can improve the bounds if $\|S(t)\|$ has sub-linear growth.
- Similar result for spaces with type p and cotype q with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, but this requires an R-bounded version of resolvent growth.

Examples

Variation of classical Example of Zabczyk 1975 and Wrobel 1989:

Example (Optimality)

Let $X = \ell^2$. $\forall \beta \in (0, \infty)$ and $\varepsilon \in (0, 1)$, $\exists A$ with resolvent growth β s.t.

$$\|S(\cdot)A^{-\tau}\| \sim \text{exponential growth} \quad \forall \tau \in [0, (1-\varepsilon)\beta)$$
$$\|S(\cdot)A^{-\tau}\| \sim \text{polynomial decay} \quad \forall \tau \geqslant \beta.$$

Example

Fix $a \in (0, \infty)$ and $b \in (0, 1)$ with $a + b \geqslant 1$. Set $\phi(s) := s^{-a} + is^b$ for $s \in (1, \infty)$. Let $X := W^{1,2}(1, \infty)$, and let A be the multiplication operator on X associated with ϕ. Then $\sigma(A) \subseteq \mathbb{C}_+$ and $-A$ generates a C_0-semigroup. and A has resolvent growth $\frac{b-1+2a}{b}$. Moreover, $\|S(t)\| \sim t^{1-\frac{1-b}{a}}$. The theory and interpolation can be applied.

Exponential stability

$$s(-A) := \sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(-A)\},$$
$$s_0(-A) := \inf\{\omega > s(-A) \mid \sup_{\mathrm{Re}(\lambda)\geqslant\omega} \|(\lambda + A)^{-1}\| < \infty\},$$
$$\omega_\alpha(S) := \inf\{\omega \in \mathbb{R} \mid \|S(t)x\| = O(e^{\omega t}) \text{ for all } x \in D(A^\alpha)\}.$$

In the previous example of Zabczyk $\omega_0(S) = 1$ and $s(-A) = 0$.

Optimal exponential stability for

- Hilbert spaces: Gearhart 1978 and Prüss 1984: $\omega_0(A) = s_0(A)$
- general Banach spaces and Fourier type: van Neerven, Straub and Weis 1995, and Weis and Wrobel 1996
- type, cotype and R-boundedness: van Neerven 2009
- positive semigroups on L^p-spaces: Weis 1995

These statement follow from our results.

New sharp result for positive semigroups

Theorem (RV, 2017c)

Let $1 \leqslant p \leqslant q < \infty$ and let X be a p-convex and q-concave Banach lattice. Let $-A$ generate a positive C_0-semigroup on X. Then

$$\omega_{\frac{1}{p}-\frac{1}{q}}(T) \leqslant s_0(-A).$$

Result is sharp in $L^p \cap L^q$.
Proof based on special multiplier result in the case $\check{m}$ is positive:

Theorem (RV, 2017a)

Let $p, q \in [1, \infty)$ with $p \leqslant q$, and let $\alpha = \frac{1}{p} - \frac{1}{q}$. Assume X is p-convex and Y is q-concave. Suppose that $\check{m}(t)$ is a positive operator for each $t \in \mathbb{R}^1$, and $t \mapsto \check{m}(t)x \in L^1(\mathbb{R}; Y)$ for all $x \in X$. Then

$$\|T_m\|_{\mathcal{L}(\dot{H}^{\alpha}_p(\mathbb{R};X),L^q(\mathbb{R};Y))} \leqslant C\|m(0)\|_{\mathcal{L}(X,Y)} \tag{3}$$

Polynomial growth

Theorem (RV, 2017c)

Let $S(t) = e^{-tA}$ be a C_0-semigroup on a Hilbert space X and assume that $\mathbb{C}_- \subseteq \rho(A)$. For $\alpha \in [1, \infty)$ and $\beta \in [0, \infty)$ consider:

(i) *There is a constant C_α such that*

$$\|(a + i\xi + A)^{-1}\| \leqslant C_\alpha\Big(\frac{1}{a^\alpha} + 1\Big), \quad a > 0,\ \xi \in \mathbb{R}$$

(ii) *There is a constant C_β such that*

$$\|S(t)\| \leqslant C_\beta\big(1 + t^\beta\big), \quad t \geqslant 0.$$

Then (i) $\Rightarrow$ (ii) *with $\beta = \alpha$ and* (ii) $\Rightarrow$ (i) *with $\alpha = \beta + 1$.*

Previously Eisner–Zwart 2006: (i) $\Rightarrow$ (ii) with $\beta = 2\alpha - 1$.
Sharpness is unknown. (ii) $\Rightarrow$ (i) with $\alpha = \beta + 1$ is sharp.
We obtained the same theorem for positive semigroups on L^p.

Sharpness in an example of Renardy

Example

On the two-dimensional torus $\mathbb{T}^2$ consider the perturbed wave equation

$$u_{tt} = u_{xx} + u_{yy} + e^{iy} u_x, \quad t > 0, x, y \in \mathbb{T},$$

This gives a Cauchy problem on $X = H^1(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$.
Renardy 1994 proved $\sigma(A) \subseteq i\mathbb{R}$, so $s_0(-A) = 0$, but $S(t) = e^{-tA}$ satisfies $\omega_0(S) \geqslant 1/2$.
We proved $\|(a + i\xi + A)^{-1}\| \leqslant C\big(\frac{1}{a} + 1\big)$. Therefore, our theorem with $\alpha = 1$ gives $\|S(t)\| \leqslant C(1 + t)e^{t/2}$ for $t \geqslant 0$ and hence $\omega_0(S) = \frac{1}{2}$.

It is an open problem whether the linear factor can be removed.

Remarks and open problems

- Conclusion: there is an analog of the theory of polynomial stability for unbounded semigroups
- In the paper we also allow $0 \in \sigma(A)$ if A is sectorial and injective
- Some cases are known to be sharp
- (L^p, L^q) Fourier multipliers are everywhere

Open problems: (see previous slides)

- Other rates than polynomial rates?
- $s_0(A) = \omega_0(S)$ for $X = L^p$ with $p \in (1, \infty) \setminus \{2\}$?
- Is there an X-valued analogue of Pitt's inequality for $1 < p \leqslant q < \infty$ and $\frac{d}{p} + \frac{d}{q} + \beta - \alpha = d$:

$$\|\xi \mapsto |\xi|^{-\alpha} \widehat{f}(\xi)\|_{L^q(\mathbb{R}^d)} \leqslant C \|s \mapsto |s|^{\beta} f(s)\|_{L^p(\mathbb{R}^d)} \quad ?$$

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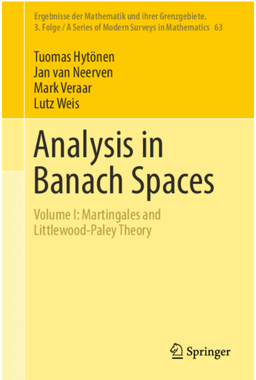

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Tuomas Hytönen, Jan van Neerven, Mark Veraar, Lutz Weis, 2016

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