Entropy and approximation numbers of Hardy integral operator in weighted spaces of Besov and Triebel–Lizorkin type

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[NU] Nasyrova M.G., Ushakova E.P. Wavelet bases and entropy numbers of Hardy operator. *Analysis Mathematica* (accepted)

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- For $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ we consider the Hardy integral operator

$$Hf(x) := \chi_{(0,\infty)}(x) \int_0^x f(y)\,dy \qquad (1)$$

in weighted Besov $B^s_{pq}(\mathbb{R}, w)$ and Triebel–Lizorkin $F^s_{pq}(\mathbb{R}, w)$ spaces on $\mathbb{R}$ with some $0 < p, q \le \infty$ and $-\infty < s < +\infty$, and admissible weights w.
The problem: Estimates on characteristic numbers (e.g. entropy, approximation etc.) of the operator $H\colon A^{s_1}_{p_1 q_1}(\mathbb{R}, w_1) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}, w_2)$, where $A^s_{pq}(\mathbb{R}, w)$ stands both for $B^s_{pq}(\mathbb{R}, w)$ and $F^s_{pq}(\mathbb{R}, w)$.

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- In particular, for $\alpha > 1 + 1/p_2$ and $\delta = s_1 - 1/p_1 - (s_2 - 1 - 1/p_2)$, we find upper estimates on $e_k(H) := e_k(H\colon A^{s_1}_{p_1q_1}(\mathbb{R}) \hookrightarrow A^{s_2}_{p_2q_2}(\mathbb{R}, w_\alpha^{-1}))$:

$$e_k(H) \lesssim k^{-(s_1 - s_2 + 1)}, \qquad k \in \mathbb{N}, \qquad \text{for } \delta + 1 < \alpha; \tag{2}$$

$$e_k(H) \lesssim k^{-(\alpha - 1 + 1/p_1 - 1/p_2)}, \qquad k \in \mathbb{N}, \qquad \text{if } \alpha < \delta + 1. \tag{3}$$

Here $w_\alpha(x) := (1 + |x|^2)^{\alpha/2}$, $0 < q_1, q_2 \leq \infty$ and p_i, s_i, $i = 1, 2$ are s.t.

$$\frac{1}{2} < p_1 < \infty,\ \ 1 < p_2 < \infty,\ \ \frac{1}{p_1} < s_1 < 1 + \min\left\{1, \frac{1}{p_1}\right\},\ \ -1 + \frac{1}{p_2} < s_2 < \frac{1}{p_2}.$$

Basic definitions: entropy and approximation numbers

[1] B. Carl, I. Stephani, Entropy, Compactness and the Approximation of Operators, Cambridge University Press, Cambridge, UK, 1990.
[2] A. Pietsch, Eigenvalues and s-numbers, Akad. Verl. G. & Portig, Leipzig, 1987.

- For $k \in \mathbb{N}$ and a continuous operator T between quasi–Banach spaces A_1 and A_2 the k–th (dyadic) entropy number $e_k(T)$ of $T\colon A_1 \to A_2$ is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover $T(U_1)$, where U_1 is the unit ball in A_1

$$e_k(T) = \inf\Big\{\varepsilon > 0\colon T(U_1) \subset \cup_{j=1}^{2^{k-1}} B(Tx_j, \varepsilon),\ x_1, \ldots, x_{2^{k-1}} \in U_1\Big\}; \quad (4)$$

$$|\lambda_k(T)| \le \left(\prod_{j=1}^{k} |\lambda_j(T)|\right)^{1/k} \le \sqrt{2} e_k(T) \qquad (k \in \mathbb{N}). \quad (5)$$

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- For $k \in \mathbb{N}$ and a linear and continuous operator T between quasi–Banach spaces A_1 and A_2 the k–th approximation number $a_k(T)$ of $T\colon A_1 \to A_2$ is the infimum of all numbers $\|T - L\|$, where L runs through the collection of all continuous linear maps from A_1 to A_2 with $\operatorname{rank} L < k$

$$a_k(T) = \inf\Big\{\|T - L\|\colon\ L : A_1 \to A_2, \operatorname{rank} L < k\Big\}; \quad (6)$$

$$\sup_{k\in\mathbb{N}} k^\alpha e_k(T) \le \sup_{k\in\mathbb{N}} k^\alpha a_k(T) \qquad (\alpha > 0). \quad (7)$$

Basic definitions: admissible weights

[3] D. Haroske, H. Triebel, Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators, I, Math. Nachr., 167 (1994), 131-156.
[4] D. Haroske, H. Triebel, Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators, II, Math. Nachr., 168 (1994), 109-137.

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\gamma \in \mathbb{N}_0$ we shall use the symbol D^γ for derivatives.

Definition 1. The class $\mathcal{W}$ of *admissible weight functions* is the collection of all infinitely differentiable functions $w\colon \mathbb{R} \to (0,\infty)$ with the following properties:
(i) for all $\gamma \in \mathbb{N}_0$ there exists a positive constant c_γ with

$$|D^\gamma w(x)| \le c_\gamma w(x) \quad \text{for all } x \in \mathbb{R}; \tag{8}$$

(ii) there exist two constants $c > 0$ and $\alpha \ge 0$ such that

$$0 < w(x) \le c\, w(y)(1 + |x-y|^2)^{\alpha/2} \quad \text{for all } x, y \in \mathbb{R}. \tag{9}$$

Remark. If w is admissible, then $w^{-1} := 1/w$ is admissible, too.
We are interested in the polynomial weight $w_\alpha \in \mathcal{W}$ of the form

$$w_\alpha(x) := (1 + |x|^2)^{\alpha/2}, \qquad x \in \mathbb{R}, \quad \alpha \ge 0. \tag{10}$$

Basic definitions: $B^s_{pq}(\mathbb{R},w)$ and $F^s_{pq}(\mathbb{R},w)$ spaces

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all complex–valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}$. By $\mathcal{S}'(\mathbb{R})$ we denote its topological dual, the space of tempered distributions on $\mathbb{R}$.

Definition 2. Let $w \in \mathcal{W}$ be an admissible weight. Let $s \in \mathbb{R}$ and $0 < q \le \infty$.
(i) Let $0 < p \le \infty$. The $B^s_{pq}(\mathbb{R},w)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{B^s_{pq}(\mathbb{R},w)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\|F^{-1}(\varphi_j \widehat{f})\right\|^q_{L^p(\mathbb{R},w)}\right)^{1/q} \tag{11}$$

(with the usual modification if $q = \infty$) is finite.
(ii) Let $0 < p < \infty$. The $F^s_{pq}(\mathbb{R},w)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{F^s_{pq}(\mathbb{R},w)} = \left\|\left(\sum_{j=0}^{\infty} 2^{jsq} |F^{-1}(\varphi_j \widehat{f})(\cdot)|^q\right)^{1/q}\right\|_{L^p(\mathbb{R},w)} \tag{12}$$

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If $s > \max\{0, 1/p-1\}$ then one can define $B^s_{pq}(\mathbb{R},w)$ as follows. For $M \in \mathbb{N}$, an admissible weight w and $f \in L^p(\mathbb{R},w)$ put $(\Delta^1_h f)(\cdot) := f(\cdot + h) - f(\cdot)$, $(\Delta^M_h f) := \Delta^1_h(\Delta^{M-1}_h f)$ and

$$\omega_M(f,t,w)_p := \sup_{|h|<t} \|\Delta^M_h f\|_{L^p(\mathbb{R},w)}, \qquad t > 0.$$

Let $\max\{0, 1/p-1\} < s < M$ and $0 < p, q \le \infty$. Then $f \in B^s_{pq}(\mathbb{R},w)$ if and only if $f \in L^p(\mathbb{R},w)$ and

$$\left(\int_0^1 \left[t^{-s}\omega_M(f,t,w)_p\right]^q \frac{dt}{t}\right)^{1/q} < \infty.$$

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[5] H. Triebel, Theory of Function Spaces, Birkhauser Verlag, Basel, 1983.

[6] H. Triebel, Theory of Function Spaces II, Birkhauser Verlag, Basel, 1992.

[7] D.E. Edmunds, H. Triebel, Function spaces, entropy numbers, differential operators, Cambridge University Press, 1996.

[8] D. Haroske, H. Triebel, Wavelet bases and entropy numbers in weighted function spaces, Math. Nachr., 278 (2005), 108-132.

[9] T. Kühn, H. Leopold, W. Sickel and L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces, I, Constr. Approx., 23 (2006), 61-77.

The history of the problem: for the operator H in Lebesgue spaces

[10] M.A. Lifshits, W. Linde, Approximation and entropy numbers of Volterra operators with application to Brownian motion, Mem. Amer. Math. Soc., 157:745 (2002), 1-87.

[11] J. Lang, Improved Estimates for the Approximation Numbers of Hardy-Type Operator, J. Approx. Theory, (1) 121 (2003), 61-70.

[12] J. Lang, A. Nekvinda, O. Mendez, Asymptotic behavior of the approximation numbers of the Hardy-type operator from L^p into L^q (case $1 < p \leq q \leq 2$ or $2 \leq p \leq q < \infty$), J. Inequal. Pure Appl. Math., (1) 5 (2004), Article 18.

[13] D.E. Edmunds, J. Lang, Approximation numbers and Kolmogorov widths of Hardy-type operators in a non-homogeneous case, Math. Nachr., (7) 279 (2006), 727-742.

[14] J. Lang, Estimates for n-widths of the Hardy-type operators, J. Approx. Theory, 140 (2006), 141-146.

[15] D.E. Edmunds, J. Lang, Operators of Hardy type, J. Comput. Appl. Math., 208 (2007), 20-28.

[16] E.N. Lomakina, V.D. Stepanov, Asymptotic estimates for the approximation and entropy numbers of a one-weight Riemann-Liouville operator, Siberian Adv. Math., (1) 17 (2007), 1-36.

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- If $1 < p < \infty$ and $w \in \mathcal{W}$ then $F^0_{p,2}((0,\infty), w)$ coincides with $L^p((0,\infty), w)$.

  Corollary 2.5 from [10]. Under assumptions of our work:

$$\|w_\alpha^{-1}\|_r \lesssim \liminf_{k\to\infty} k \cdot e_k(H : F^0_{p_1,2}((0,\infty)) \hookrightarrow F^0_{p_2,2}((0,\infty), w_\alpha^{-1})) \quad (13)$$

$$\le \limsup_{k\to\infty} k \cdot e_k(H : F^0_{p_1,2}((0,\infty)) \hookrightarrow F^0_{p_2,2}((0,\infty), w_\alpha^{-1})) \lesssim \|w_\alpha^{-1}\|_r,$$

  where $\alpha > 1 - 1/p_1 + 1/p_2 =: 1/r > 0$ and $\|w_\alpha^{-1}\|_r = \left(\int_0^\infty w_\alpha^{-r}\right)^{1/r} \simeq 1.$

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  where $\alpha > 1 - 1/p_1 + 1/p_2 =: 1/r > 0$ and $\|w_\alpha^{-1}\|_r = \left(\int_0^\infty w_\alpha^{-r}\right)^{1/r} \simeq 1$.
- Let id denote the identical operator. From Corollary 2.5 and by **[8]**

$$e_k(H : F^0_{p_1,2}((0,\infty)) \hookrightarrow F^0_{p_2,2}((0,\infty), w_\alpha^{-1})) \sim \frac{1}{k} \sim$$
$$\sim \frac{1}{k} \sim e_k(id : F^1_{p_1,2}(\mathbb{R}, w_\alpha) \hookrightarrow F^0_{p_2,2}(\mathbb{R})). \quad (14)$$

[8] D. Haroske, H. Triebel, Wavelet bases and entropy numbers in weighted function spaces, Math. Nachr., 278 (2005), 108-132.

Our approach to the solution

Part II: Reduction of the initial problem for the operator H in function spaces to some transformation $\tilde{H}$ in sequence spaces by well-developed concept of wavelet basis in $A^s_{pq}(\mathbb{R}, w)$ (atomic decomposition theorems)

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– intermediate procedure: synchronization

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Part I: Choosing two particular wavelet basis:
– $\mathrm{bas}(A_1)$ for the source space $A_1 := A^{s_1}_{p_1 q_1}(\mathbb{R})$
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specially related to each other: result of integration of elements from $\mathrm{bas}(A_2)$ must be somehow related with elements from $\mathrm{bas}(A_1)$:

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Part I: cardinal B–splines

Let $B_0 = \chi_{[0,1)}$. For $n \in \mathbb{N}$ the $n-$th order B–spline is defined recursively by

$$B_n(x) := (B_{n-1} * B_0)(x) = \frac{x}{n} B_{n-1}(x) + \frac{n+1-x}{n} B_{n-1}(x-1) \qquad (15)$$

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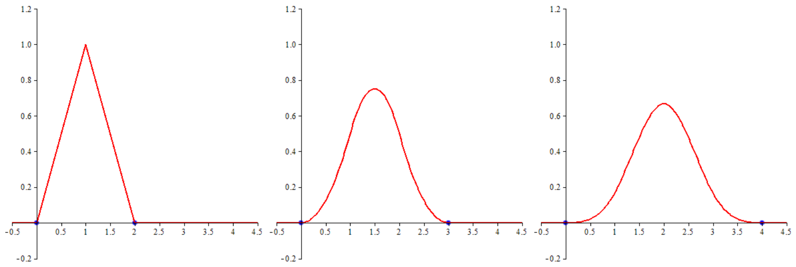

$$\operatorname{supp} B_n = [0, n+1] \qquad (n \in \mathbb{N}_0) \qquad (16)$$

$$\hat{B}_n(\omega) = [\hat{B}_0(\omega)]^{n+1} = \mathrm{e}^{-i(n+1)\omega/2} \left(\frac{\sin(\omega/2)}{\omega/2} \right)^{n+1} \qquad (17)$$

$$B_n'(x) = B_{n-1}(x) - B_{n-1}(x-1) \qquad \text{for a.a. } x \in \mathbb{R} \qquad (18)$$

[17] C.K. Chui, An Introduction to Wavelets, NY: Academic Press, 1992.

Part I: multiresolution analysis of $L^2(\mathbb{R})$ generated by B_n, $n \in \mathbb{N}$

Fixed $n \in \mathbb{N}$ and any $d, \tau \in \mathbb{Z}$ put $B_{n;\,d,\tau}(x) := B_n(2^d x - \tau)$, $x \in \mathbb{R}$. Each $d \in \mathbb{Z}$ let V_d denote the $L^2(\mathbb{R})-$closure of the linear span of the system $\{B_{n;\,d,\tau} \colon \tau \in \mathbb{Z}\}$. The spline spaces V_d, $d \in \mathbb{Z}$, which are generated by the *scaling function* B_n, constitute a multiresolution analysis of $L^2(\mathbb{R})$ in the sense that

- $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$;
- $\mathrm{clos}_{L^2(\mathbb{R})}\Big(\bigcup_{d\in\mathbb{Z}} V_d\Big) = L^2(\mathbb{R})$;
- $\bigcap_{d\in\mathbb{Z}} V_d = \{0\}$;
- for each d the $\{B_{n;\,d,\tau} \colon \tau \in \mathbb{Z}\}$ is an unconditional (but not orthonormal) basis of V_d.

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There are the orthogonal complementary subspaces $\ldots, W_{-1}, W_0, W_1, \ldots$:

- $V_{d+1} = V_d \oplus W_d$ for all $d \in \mathbb{Z}$, where $\oplus$ stands for $V_d \perp W_d$ and $V_{d+1} = V_d + W_d$.

Wavelet subspaces W_d, $d \in \mathbb{Z}$, related to the spline B_n, are also generated by some basic functions (*wavelets*) in the same manner as that the spline spaces V_d, $d \in \mathbb{Z}$, are generated by the spline B_n.

Part I: the Battle–Lemarié family

[18] G. Battle, A block spin construction of ondelettes, Part I: Lemarie functions, Comm. Math. Phys., 110 (1987), 601-615.
[19] P.G. Lemarie, Une nouvelle base d'ondelettes de $L^2(\mathbb{R}^n)$, J. de Math. Pures et Appl., 67 (1988), 227-236.

- The function $\phi_n^{BL} \in L^2(\mathbb{R})$ satisfying

$$\hat{\phi}_n^{BL}(\omega) = \hat{B}_n(\omega)\left(\sum_{m\in\mathbb{Z}} \left|\hat{B}_n(\omega + 2\pi m)\right|^2\right)^{-1/2} \tag{19}$$

is called the Battle–Lemarié scaling function. Integer translations of ϕ_n^{BL} form an orthonormal basis in V_0 of the MRA generated by B_n.

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- The $n-$th order Battle-Lemarie wavelet is the function ψ_n^{BL} with

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  Integer translations of ψ_n^{BL} form an orthonormal basis in W_0 of the multiresolution analysis generated by B_n.

- A scaling function ϕ and one of its associated wavelets ψ form a wavelet system $\{\phi, \psi\}$.

Part I: properties of Battle–Lemarié wavelet systems

- for $t \in \mathbb{R}$

$$\phi_n^{BL}(t) = \sum_{k\in\mathbb{Z}} \alpha_k B_n(t-k), \quad \psi_n^{BL}(t) = \sum_{k\in\mathbb{Z}} \beta_k B_n(2t-k), \quad n \in \mathbb{N}. \quad (21)$$

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- the system $\left\{2^{d/2} h^n_{d\tau} : d \in \mathbb{N}_{-1},\ \tau \in \mathbb{Z}\right\}$, $\mathbb{N}_{-1} := \mathbb{N} \cup \{-1, 0\}$, where

$$h^n_{-1\tau}(x) := \sqrt{2}\phi_n^{BL}(x-\tau) \qquad \text{and} \qquad h^n_{d\tau}(x) := \psi_n^{BL}(2^d x - \tau), \quad (22)$$

is an orthonormal basis in $L^2(\mathbb{R})$ with the following properties:
– ϕ_n^{BL} and ψ_n^{BL} have continuous derivatives up to order $n-1$ on $\mathbb{R}$;
– the restriction of ϕ_n^{BL} and ψ_n^{BL} to each interval $(k, k+\frac{1}{2})$ with $2k \in \mathbb{Z}$ is a polynomial of degree at most n;
– there are constants $c > 0$ and $\alpha > 0$ such that for $m = 0, 1, \ldots, n$

$$\left|\frac{d^m}{dx^m}\phi_n^{BL}(x)\right| + \left|\frac{d^m}{dx^m}\psi_n^{BL}(x)\right| \le c \cdot \mathrm{e}^{-\alpha|x|}, \qquad 2x \in \mathbb{R} \setminus \mathbb{Z};$$

– for $m = 0, 1, \ldots, n$ it holds: $\int_{\mathbb{R}} x^m\, \psi_n^{BL}(x)\, dx = 0$;

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- supports of ϕ_n^{BL} and ψ_n^{BL} are not compact on $\mathbb{R}$ for $n \in \mathbb{N}$.

Part I & synchronization: two complementary questions

I. Explicit formulae for $\{\phi_n^{BL}, \psi_n^{BL}\}$:

$$\phi_n^{BL}(t) = \sum_{k\in\mathbb{Z}} \alpha_k B_n(t-k), \quad \psi_n^{BL}(t) = \sum_{k\in\mathbb{Z}} \beta_k B_n(2t-k), \quad n \in \mathbb{N}; \tag{23}$$

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$$\Phi_n^{BL}(t) = \sum_{k=1}^{M_\phi(n)} \gamma_k(n)\, \phi_n^{BL}(t-k), \qquad t \in \mathbb{R}, \tag{24}$$

$$\Psi_n^{BL}(t) = \sum_{k=1}^{M_\psi(n)} s_k(n)\, \psi_n^{BL}(t-k/2), \qquad t \in \mathbb{R}, \tag{25}$$

where Φ_n^{BL} and Ψ_n^{BL} have compact supports on $\mathbb{R}$, and, up to a constant, coefficients $\gamma_k(n)$ (or $s_k(n)$) are related to $\gamma_k(n-1)$ and $\gamma_k(n+1)$ (or to $s_k(n-1)$ and $s_k(n+1)$, respectively).

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[UU] Ushakova E.P., Ushakova K.E. Localisation property of Battle–Lemarié wavelets' sums. (https://arxiv.org/abs/1708.09536)

[20] Ya. Novikov, S.B. Stechkin, Basic wavelet theory, Russian Math. Surveys 53:6 (1998), 1159-1231.
[21] P. Wojtaszczyk, A mathematical introduction to wavelets, Cambridge University Press, Cambridge, UK, 1997.

Part I & synchr.: construction and localisation of $\{\phi_n^{BL}, \psi_n^{BL}\}$, $n = 1$

[20] Ya. Novikov, S.B. Stechkin, Basic wavelet theory, Russian Math. Surveys 53:6 (1998), 1159-1231.
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For $$B_1(x) = \begin{cases} x, & \text{if} \quad 0 \le x < 1, \\ 2 - x, & \text{if} \quad 1 \le x < 2, \\ 0 & \text{otherwise}; \end{cases}$$

$$\hat{\phi}_1^{BL}(\omega) = \hat{B}_1(\omega) \left(\sum_{m \in \mathbb{Z}} \left| \hat{B}_1(\omega + 2\pi m) \right|^2 \right)^{-1/2} =: \hat{B}_1(\omega) \mathbb{P}_1(\omega)^{-1/2}; \quad (26)$$

$$\mathbb{P}_1(\omega) = \frac{1}{6t} |\mathrm{e}^{\pm i\omega} t + 1|^2, \qquad t = r^{\pm 1}, \quad r := 2 - \sqrt{3}; \quad (27)$$

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$$\frac{1}{\mathrm{e}^{\pm i\omega} r + 1} = \sum_{l=0}^{\infty} \left(-r\, \mathrm{e}^{\pm i\omega} \right)^l, \qquad j = 1, \ldots, n. \quad (28)$$

Therefore, since for $r = 2 - \sqrt{3}$ and $\beta = \sqrt{6(2-\sqrt{3})}$,

$$\hat{\phi}_1^{\pm}(\omega) = \frac{\beta\ \hat{B}_1(\omega)}{\mathrm{e}^{\pm i\omega} r + 1}, \quad \text{then} \quad \phi_1^{\pm}(x) = \beta \sum_{l=0}^{\infty} (-r)^l B_1(x \pm l). \tag{29}$$

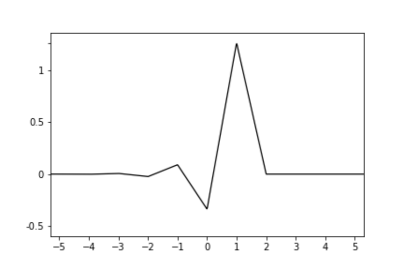

Figure: ϕ_1^+

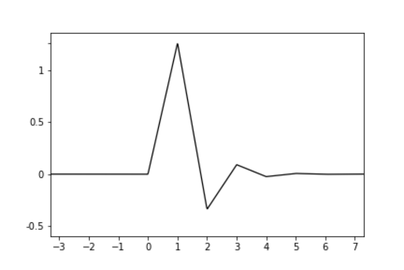

Figure: ϕ_1^-

In order to obtain a localized function Φ_1^+, we have to do the following:

$$\Phi_1^+(\cdot) := \phi_1^+(\cdot) + r\phi_1^+(\cdot + 1) = \beta B_1(\cdot). \tag{30}$$

Part I & synchr.: construction and localisation of $\{\phi_n^{BL}, \psi_n^{BL}\}$, $n = 1$

The $\hat{\psi}_1^{\pm}$ and one of $\psi_1^{\pm} = \psi_t^{\pm}$ (for $t = r$) related to $\phi_1^{\pm}$ have the forms:

$$\hat{\psi}_1^{\pm}(\omega) = \frac{\beta}{4} \; \frac{[1 - t\mathrm{e}^{\mp i\omega/2}]\,[\mathrm{e}^{i\omega/2} - 2 + \mathrm{e}^{-i\omega/2}]}{[\mathrm{e}^{\mp i\omega} t + 1][\mathrm{e}^{\pm i\omega/2} t + 1]} \hat{B}_1(\omega/2) =: \hat{\psi}_t^{\pm}(\omega); \quad (31)$$

$$\psi_r^{\pm}(x) = \frac{r\beta}{4} \sum_{k \geq 0} (-r)^k \sum_{m \geq 0} (-r)^m \Big[\frac{1}{r} B_1(2x \mp 2k \pm m + 1) \quad (32)$$

$$-(1 + \frac{2}{r}) B_1(2x \mp 2k \pm m) + (2 + \frac{1}{r}) B_1(2x \mp 2k \pm m \mp 1) - B_1(2x \mp 2k \pm m \mp 2)\Big].$$

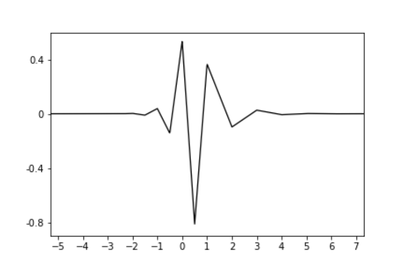

Figure: ψ_r^+

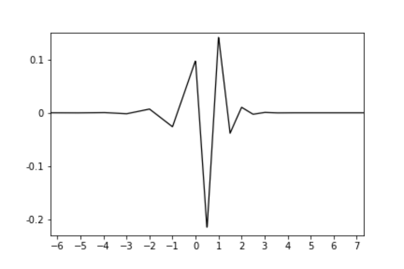

Figure: $\psi_{1/r}^+$

Since

$$\hat{\psi}_t^{\pm}(\omega) \simeq \begin{cases} \dfrac{[1/r - \mathrm{e}^{\mp i\omega/2}]\,[\mathrm{e}^{i\omega/2} - 2 + \mathrm{e}^{-i\omega/2}]}{[\mathrm{e}^{\mp i\omega} r + 1][\mathrm{e}^{\pm i\omega/2} r + 1]} \hat{B}_1(\omega/2), & t = r \\ \dfrac{[r - \mathrm{e}^{\mp i\omega/2}]\,[\mathrm{e}^{i\omega/2} - 2 + \mathrm{e}^{-i\omega/2}]}{[\mathrm{e}^{\pm i\omega} r + 1][\mathrm{e}^{\mp i\omega/2} r + 1]}\ \mathrm{e}^{\pm i\omega/2}\ \hat{B}_1(\omega/2), & t = \frac{1}{r} \end{cases} \tag{33}$$

then, in order to obtain a localized function Ψ_1^+, we, firstly, do as follows:

$$\Lambda_t^+(\cdot) := \psi_t^+(\cdot - \frac{1}{2}) + r[\psi_1^+(\cdot) + \psi_1^+(\cdot - \frac{3}{2})] + r^2\psi_1^+(\cdot - 1). \tag{34}$$

Secondly, after (34), we finalize the localisation procedure by

$$\Psi_1^+(\cdot) = \Lambda_r^+(\cdot) - \Lambda_{1/r}^+(\cdot - \frac{1}{2}) \simeq B_1(2\cdot) - 2B_1(2\cdot -1) + B_1(2\cdot -2) = B_3^{(2)}(2\cdot), \tag{35}$$

which corresponds to $[\mathrm{e}^{i\omega/2} - 2 + \mathrm{e}^{-i\omega/2}]\hat{B}_1(\omega/2)$ in (33).

Part II: atomic decomposition theorems in $A^{s_1}_{p_1q_1}(\mathbb{R})$ and $A^{s_2}_{p_2q_2}(\mathbb{R}, w_\alpha^{-1})$

Let χ_{jk} be the characteristic function of $I_{jk} := \left[2^{-j}k, 2^{-j}(k+1)\right)$, where $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, and let for $0 < p \le \infty$ $\chi^{(p)}_{jk}(x) := 2^{j/p}\chi_{jk}(x)$.

Proposition [22, Thm. 9.1 & Rem. 9.1]. *Let $1 < p < \infty$, $w \in \mathcal{W}$ and $\{\phi^0, \psi^0\}$ be the Haar system. If $0 < q \le \infty$ and $\frac{1}{p} - 1 < s < \frac{1}{p}$, then a distribution $g \in \mathcal{S}'(\mathbb{R})$ belongs to $B^s_{pq}(\mathbb{R}, w)$ if and only if*

$$\|g\|^{\diamond}_{B^s_{pq}(\mathbb{R},w)} := \left(\int_{\mathbb{R}}\left|\sum_{k\in\mathbb{Z}}\langle g, \phi^0_k\rangle \chi_{0k}(x)\right|^p w^p(x)\,dx\right)^{\frac{1}{p}}$$

$$+\left(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p}+1)q}\left(\int_{\mathbb{R}}\left|\sum_{k\in\mathbb{Z}}\langle g, \psi^0_{jk}\rangle \chi^{(p)}_{jk}(x)\right|^p w^p(x)\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty$$

with the usual modification if $q = \infty$. The $\|g\|^{\diamond}_{B^s_{pq}(\mathbb{R},w)}$ may be used as an equivalent norm on $B^s_{pq}(\mathbb{R}, w)$.

[22] A. Malecka, Haar functions in weighted Besov and Triebel–Lizorkin spaces, J. Approx. Theory, 200 (2015), 1-27.

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- Since $g = Hf$ and, therefore, $g = 0$ on $\mathbb{R}^-$, then $\|g\|^{\diamond}_{B^s_{pq}((0,\infty),w)} \le \|\widetilde{g}\|^{\diamond}_{B^s_{pq}(\mathbb{R},w)}$, where $\widetilde{g}$ is an extension of g to $\mathbb{R}$.

Part II: atomic decomposition theorems in $A^{s_1}_{p_1q_1}(\mathbb{R})$ and $A^{s_2}_{p_2q_2}(\mathbb{R},w_\alpha^{-1})$

Proposition [23, Thms. 2.46 & 2.49]. *Let* $0<p,q\le\infty$ *and* $\{\phi^1,\psi^1\}$ *be the Battle–Lemarié system of the 1st order. For* $\max\{\frac{1}{p},1\}-2<s<1+\min\{\frac{1}{p},1\}$ *a distribution* $f\in\mathcal{S}'(\mathbb{R})$ *belongs to* $B^s_{pq}(\mathbb{R})$ *if and only if it can be represented as*

$$f=\sum_{k\in\mathbb{Z}}\langle f,\phi^1_k\rangle\phi^1_k+\sum_{j\in\mathbb{N}_0}2^j\sum_{k\in\mathbb{Z}}\langle f,\psi^1_{jk}\rangle\psi^1_{jk}, \tag{36}$$

uncond. converg. being in $\mathcal{S}'(\mathbb{R})$ *and loc. in any space* $B^\sigma_{pq}(\mathbb{R})$ *for* $\sigma<s$, *where*

$$\|f\|^\star_{B^s_{pq}(\mathbb{R})}:=\Big(\sum_{k\in\mathbb{Z}}\Big|\langle f,\phi^1_k\rangle\Big|^p\Big)^{\frac{1}{p}}+\Big(\sum_{j=0}^\infty 2^{j(s-\frac{1}{p}+1)q}\Big(\sum_{k\in\mathbb{Z}}\Big|\langle f,\psi^1_{jk}\rangle\Big|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}<\infty \tag{37}$$

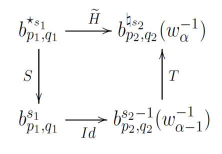
with usual modifications for $p=\infty$ *and* $q=\infty$. *Moreover,* $\|f\|^\star_{B^s_{pq}(\mathbb{R})}$ *is equivalent to the norm on* $B^s_{pq}(\mathbb{R})$.

[23] H. Triebel, Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration, European Math. Soc. Publishing House, Zurich, 2010.

Remark [UU]. By localisation property of Battle–Lemarié systems, $\|f\|^\star_{B^s_{pq}(\mathbb{R})}\approx$

$$\Big(\sum_{k\in\mathbb{Z}}\Big|\langle f,B_1(\cdot-k)\rangle\Big|^p\Big)^{1/p}+\Big(\sum_{j=0}^\infty 2^{j(s-\frac{1}{p}+1)q}\Big(\sum_{k\in\mathbb{Z}}\Big|\langle f,B_3^{(2)}(2(2^j\cdot-k))\rangle\Big|^p\Big)^{q/p}\Big)^{1/q}.$$

Part III: Transition from $\tilde{H}$ to id in seq. spaces $a^{s_1}_{p_1 q_1}(w_{\alpha-1})$ and $a^{s_2-1}_{p_2 q_2}$



$\frac{1}{2} < p_1 < \infty,\ 1 < p_2 < \infty,$

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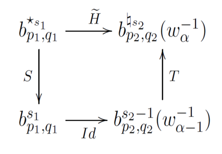
Part III: Transition from $\widetilde{H}$ to id in seq. spaces $a^{s_1}_{p_1 q_1}(w_{\alpha-1})$ and $a^{s_2-1}_{p_2 q_2}$



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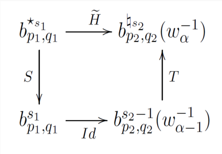
- $e_n(Id\colon b^{s_1}_{p_1 q_1} \to b^{s_2-1}_{p_2 q_2}(w^{-1}_{\alpha-1})) \approx e_n(id\colon b^{s_1}_{p_1 q_1}(w_{\alpha-1}) \to b^{s_2-1}_{p_2 q_2})$,
  where the operator id is compact with $\alpha > 1$ if

$$s_2 - 1 < s_1,\ \alpha > 1 - \frac{1}{p_1} + \frac{1}{p_2},\ \delta = s_1 - \frac{1}{p_1} - \left(s_2 - 1 - \frac{1}{p_2}\right) > 0. \quad (39)$$

Part III: Transition from $\tilde{H}$ to id in seq. spaces $a^{s_1}_{p_1 q_1}(w_{\alpha-1})$ and $a^{s_2-1}_{p_2 q_2}$



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$$s_2 - 1 < s_1,\ \alpha > 1 - \frac{1}{p_1} + \frac{1}{p_2},\ \delta = s_1 - \frac{1}{p_1} - \left(s_2 - 1 - \frac{1}{p_2}\right) > 0. \quad (39)$$

By multiplicativity properties of entropy and approximation numbers of

$$\widetilde{H} = T \circ Id \circ S, \qquad \text{where } \|S\| \lesssim 1, \text{ and } \|T\| \lesssim 1, \quad (40)$$

we come to our results applying **[8]**, **[9]** and **[24]**.

[8] D. Haroske, H. Triebel, Wavelet bases and entropy numbers in weighted function spaces, Math. Nachr., 278 (2005), 108-132.
[9] T. Kühn, H. Leopold, W. Sickel and L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces, I, Constr. Approx., 23 (2006), 61-77.
[24] L. Skrzypczak, On approximation numbers of Sobolev embeddings of weighted function spaces, J. Approx. Theory, 136 (2005), 91-107.

Upper estimates on entropy numbers $e_k(H)$ of the operator H:

Theorem 1. *Let* $0 < q_1 \le \infty$, $0 < q_2 \le \infty$, $1 < p_2 < \infty$, $\alpha > 1 + 1/p_2$ *and* $1/p_2 - 1 < s_2 < 1/p_2$. *Suppose that* $1/2 < p_1 \le \infty$ *with* $1/2 < p_1 < \infty$ *in the case* $A^{s_1}_{p_1 q_1}(\mathbb{R}) = F^{s_1}_{p_1 q_1}(\mathbb{R})$ *and* $1/p_1 < s_1 < 1 + \min\{1, 1/p_1\}$. *Put*

$$\delta := s_1 - 1/p_1 - (s_2 - 1 - 1/p_2)\,.$$

Then $H : A^{s_1}_{p_1 q_1}(\mathbb{R}) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}, w_\alpha^{-1})$ *of the form* (1) *is compact. In addition,*
(i) *if* $\delta + 1 < \alpha$ *then*

$$e_k(H) \lesssim \begin{cases} k^{-(s_1 - s_2 + 1)}, & \delta + 1 < \alpha, \\ k^{-(\alpha - 1 + 1/p_1 - 1/p_2)} & \delta + 1 > \alpha, \end{cases} \qquad (k \in \mathbb{N});$$

(iii) *in the case* $\alpha - 1 = \delta$ *we have the following estimates in the case* $A^{s_i}_{p_i q_i}(\mathbb{R}, w) = B^{s_i}_{p_i q_i}(\mathbb{R}, w)$, $i = 1, 2$, *with* $\tau := s_1 - s_2 + 1 + \frac{1}{q_2} - \frac{1}{q_1}$*:*

$$e_k(H) \lesssim \begin{cases} k^{-(s_1 - s_2 + 1)} (1 + \log k)^{\tau}, & \tau > 0, \\ k^{-(s_1 - s_2 + 1)} (1 + \log\log k)^{\frac{1}{q_1}}, & \tau = 0, \\ k^{-(s_1 - s_2 + 1)}, & \tau < 0, \end{cases} \qquad (1 < k \in \mathbb{N});$$

if $\alpha - 1 = \delta$, $q_1 \le p_1$ *and* $q_2 \ge p_2$ *in the case* $A^{s_i}_{p_i q_i}(\mathbb{R}, w) = F^{s_i}_{p_i q_i}(\mathbb{R}, w)$, $i = 1, 2$, *then*

$$e_k(H) \lesssim k^{-(s_1 - s_2 + 1)} (\log k)^{\alpha - 1} \quad (1 < k \in \mathbb{N}).$$

Upper estimates on approximation numbers $a_k(H)$ of the operator H:

Theorem 2. *Let* $1 \le q_1 \le \infty$, $1 \le q_2 \le \infty$, $1 < p_2 < \infty$, $\alpha > 1 + 1/p_2$ *and* $1/p_2 - 1 < s_2 < 1/p_2$. *Suppose that* $1 \le p_1 \le \infty$ *with* $1 \le p_1 < \infty$ *in the case* $A^{s_1}_{p_1 q_1}(\mathbb{R}) = F^{s_1}_{p_1 q_1}(\mathbb{R})$ *and* $1/p_1 < s_1 < 1 + 1/p_1$. *Put*

$$\delta := s_1 - \frac{1}{p_1} - \left(s_2 - 1 - \frac{1}{p_2}\right).$$

Assume that $\bar{\alpha} := \alpha - 1 \neq \delta$. *Let* $a_n(H)$ *denote the* $n-$*th approximation number of the operator* $H : A^{s_1}_{p_1 q_1}(\mathbb{R}) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}, w_\alpha^{-1})$ *of the form* (1). *Then*

$$a_k \lesssim k^{-\varkappa},$$

where

$$\varkappa = \begin{cases} \min\{\bar{\alpha}, \delta\}, & 1 \le p_1 \le p_2 \le 2 \text{ or } 2 \le p_1 \le p_2 \le \infty \\ \min\{\bar{\alpha}, \delta\} + \frac{1}{p_1} - \frac{1}{p_2}, & p_2 < p_1 \le \infty \end{cases}$$

and $\varkappa = \min\{\bar{\alpha}, \delta\} + \frac{1}{2} - \frac{1}{\min\{p_1', p_2\}}$, *if* $1 \le p_1 < 2 < p_2 \le \infty,\ (p_1, p_2) \neq (1, \infty)$ *and* $\min\{\bar{\alpha}, \delta\} > \frac{1}{\min\{p_1', p_2\}}$; *and* $\varkappa = \min\{\bar{\alpha}, \delta\} \cdot \frac{\min\{p_1', p_2\}}{2}$, *in the case* $1 \le p_1 < 2 < p_2 \le \infty,\ (p_1, p_2) \neq (1, \infty)$ *and* $\min\{\bar{\alpha}, \delta\} \le \frac{1}{\min\{p_1', p_2\}}$.