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**Whitney-type Problems for Sobolev Spaces.**
**The case $1 < p \leqslant n$.**

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# Classical Whitney Problems

Let $S$ be an arbitrary closed subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$.

The Classical Whithey Problem 1 *How can one tell whether a given function $f$ defined on the set $S$ extends to a function $F \in C^m(\mathbb{R}^n)$? If such an $F$ exists, then how small can we take its $C^m$-norm?*

By $C^m(\mathbb{R}^n)|_S$ we shall denote the trace space of the space $C^m(\mathbb{R}^n)$ with the quotient-space semi-norm

$$\|f|C^m(\mathbb{R}^n)|_S\| := \inf \|F|C^m(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible extensions $F$ of the function $f$.

The Classical Whithey Problem 2 *Does there exist a linear bounded operator* $\mathrm{Ext} : C^m(\mathbb{R}^n)|_S \to C^m(\mathbb{R}^n)$ *which is an extension operator in the sense that* $\mathrm{Tr} \circ \mathrm{Ext}[f](x) = f(x)$ *for every* $x \in S$*?*

# Solutions to the Classical Whitney Problems

H. Whitney solved these Problems only for $C^m(\mathbb{R})$ with $m \geqslant 1$ and $C^{0,1}(\mathbb{R}^n)$ (the space of Lipschitz functions) spaces with $n \geqslant 1$ in 1934 ([1], [2]).

In the decades since Whitney's seminal work, fundamental progress was made by G. Glaeser (1958), Y. Brudnyi and P. Shvartsman (1980-2000), and E. Bierstone, P. Milman, and W. Pawlucki (2003).

C. Fefferman gave a complete solution to the above problems (2005-2007).

# Whitney-type Problems for Sobolev Spaces

Recall that, when $p > n$, $m \in \mathbb{N}$ we have $W_p^m(\mathbb{R}^n) \subset C^{m-1,1-\frac{n}{p}}(\mathbb{R}^n)$. Then any $F \in W_p^m(\mathbb{R}^n)$ has a well defined restrictions of $D^\alpha F$, $|\alpha| \leqslant m-1$ to an arbitrary subset $S \subset \mathbb{R}^n$.

Sobolev imbedding allows to formulate analogs of Classical Whitney Problems in the context of Sobolev spaces $W_p^m(\mathbb{R}^n)$, $p \in (n, \infty]$, $m \in \mathbb{N}$.

In 2014 P. Shvartsman [7] solved analogs of the Classical Whitney Problems for Sobolev spaces in the case $n = 2$, $m = 2$, $p > 2$.

C. Fefferman, A. Israel and G.K. Luli [5] solved analog of the second Classical Whitney Problem in the context of Sobolev spaces in the case $n, m \in \mathbb{N}$, $p > n$ (2014-2016).

# Whitney-type Problems for Sobolev Spaces

In the case $1 < p \leqslant n$ we need more delicate approach to formulate Whitney-type problems.

Example

We set $F_0(x) := \varphi(x) \ln(\ln(\frac{1}{|x|}))$ for sufficiently smooth cut-function $\varphi$ with compact support. Then we set

$$F(x) := \sum_{k=0}^{\infty} 2^{-k} F_0(x - r_k), \; \{r_k\}_{k=0}^{\infty} = \mathbb{Q}^n.$$

We have $F \in W_n^1(\mathbb{R}^n)$ but $F$ is essentially unbounded in the neighborhood of every point.

# Whitney-type Problems for Sobolev Spaces

Recall that in the case $d \in [0, n]$, $p \in (\max\{1, n-d\}, \infty]$ for every $F \in W_p^1(\mathbb{R}^n)$ there exist set $E_F \subset \mathbb{R}^n$ and representative $\overline{F}$, such that $\mathcal{H}^d(E_F) = 0$ and

$$\overline{F}(x) = \lim_{r \to 0} \fint_{B(x,r)} F(y)\, d\mathcal{L}_n(y), \quad \forall x \in \mathbb{R}^n \setminus E_F.$$

Moreover, every point $x \in \mathbb{R}^n \setminus E_F$ is a Lebesgue point of $\overline{F}$.

Good News: For every set $S$ with $\dim_H S \geqslant d$ we can define the trace of $F \in W_p^1(\mathbb{R}^n)$, $p \in (\max\{1, n-d\}, \infty]$ to $S$ as the pointwise restriction of $\overline{F}$ to $S$. Denote the trace of $F$ to $S$ by $F|_S$.

# Trace Space

By $W_p^1(\mathbb{R}^n)|_S$ we shall denote the trace space of the space $W_p^1(\mathbb{R}^n)$ with the quotient-space norm. More precisely, we set

$$W_p^1(\mathbb{R}^n)|_S := \{f : S \to \mathbb{R} | \exists F \in W_p^1(\mathbb{R}^n) \text{ s.t. } F|_S = f\}.$$

Equip this space with the norm

$$\|f|W_p^1(\mathbb{R}^n)|_S\| = \inf_{F|_S=f} \|F|W_p^1(\mathbb{R}^n)\|.$$

The trace operator $\mathrm{Tr}\,|_S : W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)|_S$ takes $F$ and gives back restriction of its "good representative"$\overline{F}$ to $S$.

# Whitney-type Problems for Sobolev Spaces

Problem A. *Fix parameters* $d \in [0, n]$, $p \in (\max\{1, n - d\}, \infty]$ *and a closed set* $S \subset \mathbb{R}^n$ *with* $\dim_H S \geqslant d$. *Given a function* $f : S \to \mathbb{R}$, *how can we decide whether there exists a function* $F \in W_p^1(\mathbb{R}^n)$ *such that* $F|_S = f$*? Consider the* $W_p^1(\mathbb{R}^n)$*-norm of all functions* $F \in W_p^1(\mathbb{R}^n)$ *such that* $F|_S = f$ *on* $S$. *How small can these norms be?*

We also consider the following closely related problem.

Problem B. *Fix parameters* $d \in [0, n]$, $p \in (\max\{1, n - d\}, \infty]$ *and a closed set* $S \subset \mathbb{R}^n$ *with* $\dim_H S \geqslant d$. *Is there exist a bounded linear operator* $\mathrm{Ext} : W_p^1(\mathbb{R}^n)|_S \to W_p^1(\mathbb{R}^n)$ *such that* $\mathrm{Tr}|_S \circ \mathrm{Ext} = \mathrm{Id}$ *on* $W_p^1(\mathbb{R}^n)|_S$*?*

# Whitney-type Problems for Sobolev Spaces

One can find a huge amount of papers devoted to problems A and B for Sobolev spaces in the case $p \in (1, n]$, $m \in \mathbb{N}$ (see books of V. Maz'ya for references).

Unfortunately all previously known papers dealt with "good"closed sets $S$: Ahlofrs regular sets, single cusps, etc.

What about the Problems A and B for more complicated set $S$ and $1 < p \leqslant n$?

# Hausdorff content VS Hausdorff measure

Let $0 \leqslant d \leqslant n$, $S$ be an arbitrary subset of $\mathbb{R}^n$. Consider for every $\delta \in (0, \infty]$ the following set function

$$\mathcal{H}^d_\delta(S) = \inf \sum_j r_j^d,$$

where the infimum is taken over all countable coverings of $S$ by balls $B(x_j, r_j)$ with arbitrary centres $x_j \in \mathbb{R}^n$ and radius $r_j \in (0, \delta)$.

The $d$-Hausdorff content of $S$ is defined as $\mathcal{H}^d_\infty(S)$.

The $d$-Hausdorff measure of $S$ is defined as

$$\mathcal{H}^d(S) := \lim_{\delta \to 0} \mathcal{H}^d_\delta(S).$$

## $d$-thick sets VS $d$-regular sets

Recall that a set $S \subset \mathbb{R}^n$ is Ahlfors $d$-regular iff for every $x \in S$ and $r \in (0,1]$

$$\mathcal{H}^d(Q(x,r) \cap S) \approx r^d.$$

We say that a set $S$ is $d$-thick iff for every $x \in S$ and $r \in (0,1]$

$$\mathcal{H}^d_\infty(Q(x,r) \cap S) \approx r^d.$$

# Examples od $d$-thick sets

Examples:

(1) The closure of an arbitrary domain $\overline{\Omega}$ (open and path-connected set) is 1-thick;

(2) Every Ahlfors $d$-regular set is $d$-thick. The converse is false.

(3) Every $(\varepsilon, \delta)$-domain is $n$-thick.

# Tools: $d$-regular systems of measures on $S$

Definition *Let $d \in [0,n]$. Assume that $\dim_H S \geqslant d$. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a family of Borel measures with $\operatorname{supp}\mu_k \subset S$, $k \in \mathbb{N}_0$. We say that $\{\mu_k\}_{k\in\mathbb{N}_0}$ is a d-regular system of measures on $S$ if and only if for some universal constants $C_1, C_2, C_3 > 0$ the following properties hold for every $k \in \mathbb{N}_0$:*

(1)

$$\mu_k(B(x,r)) \leqslant C_1 r^d \quad \forall x \in \mathbb{R}^n, \forall r \in (0, 2^{-k}];$$

(2)

$$\mu_k(B(x,r)) \geqslant C_2 r^d \quad \forall x \in S, \forall r \geqslant 2^{-k};$$

(3)

$$2^{d-n}\mu_k(G) \leqslant \mu_{k-1}(G) \leqslant \mu_k(G) \quad \forall \text{ Borel set } G \subset S.$$

Theorem. *Let $d \in [0,n]$. Assume that $S \subset \mathbb{R}^n$ is d-thick. Then there exists a d-regular system of measures on $S$.*

# Examples of $d$-regular systems of measures on $S$

Examples:

$(1)$ If $S$ is an Ahlfors $d$-regular set, then one can take $\mu_k = C\mathcal{H}^d$ for every $k$ with $C > 0$ independent on $k$;

$(2)$ If $S = \overline{\Omega}$ for $(\varepsilon, \delta)$-domain $\Omega$, then one can take $\mu_k = \mathcal{L}_n$ for every $k$.

# Tools:Porous subsets

Let $S$ be a closed nonempty subset of $\mathbb{R}^n$ and $\lambda \in (0,1)$. For every $j \in \mathbb{N}_0$ define

$$S_j(\lambda) := \{x \in S | \exists y \in Q(x, 2^{-j}) \text{ for which } Q(y, \lambda 2^{-j}) \subset \mathbb{R}^n \setminus S\}.$$

and call $S_j(\lambda)$ maximal $j$-porous subset of $S$. We say that $S$ *is porous* if there exists a number $\lambda \in (0,1)$ such that $S_j(\lambda) = S$ for every $j \in \mathbb{N}_0$.

Examples:

(1) Assume that $d \in (0, n)$ and $S$ is an Ahlfors $d$-regular closed set. Then $\exists \lambda$ s.t. $S_j(\lambda) = S(\lambda)$ for every $j \in \mathbb{N}_0$.

(2) Let $\Omega$ be an $(\varepsilon, \delta)$-domain. Let $S = \partial\Omega$. Then $\exists \lambda$ s.t. $S_j(\lambda) = S(\lambda)$ for every $j \in \mathbb{N}_0$.

# Tools:Calderon-type maximal functions

Let $S$ be an arbitrary closed $d$-thick set. Let $\{\mu_k\} = \{\mu_k\}_{k\in\mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n, \mu_k)$, $\forall k \in \mathbb{N}_0$. Given $t \in [0,1)$ for every $x \in S$ we set

$$f^{\sharp}_{\{\mu_k\}}(x,t) := \sup_{r\in(t,1)} \frac{1}{r} \fint_{Q(x,r)\cap S} \left| f(y) - \fint_{Q(x,r)\cap S} f(z)\, d\mu_{k(r)}(z) \right| d\mu_{k(r)}(y), \tag{1}$$

where $k(r)$ is the unique integer such that $r \in [2^{-k(r)}, 2^{-k(r)+1})$.

We set $f^{\sharp}_{\{\mu_k\}}(x) := f^{\sharp}_{\{\mu_k\}}(x,0)$.

In the case $S = \mathbb{R}^n$, $\mu_k = \mathcal{L}_n$ we obtain classical sharp-function.

# Trace norm

Let $S$ be a closed $d$-thick set for some $d \in [0, n]$. Let $p \in (1, \infty)$. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Assume that function $f \in L_1^{\mathrm{loc}}(S, \mu_k)\ \forall k \in \mathbb{N}_0$. Given $p \in (1, \infty)$, $\lambda \in (0, 1)$ we set

$$\mathcal{SN}_p[f] := \|f^{\sharp}_{\{\mu_k\}} | L_p(S, \mathcal{L}_n)\|;$$

$$\mathcal{BN}_{p,\lambda}[f] := \|f | L_p(S, \mu_0)\|$$

$$+ \Big( \sum_{j \in \mathbb{N}_0} 2^{j(d-n)} \int\limits_{S_j(\lambda)} (f^{\sharp}_{\{\mu_k\}}(x, 2^{-j}))^p \, d\mu_j(x) \Big)^{\frac{1}{p}};$$

$$\mathcal{N}_{p,\lambda}[f] := \mathcal{SN}_p[f] + \mathcal{BN}_{p,\lambda}[f].$$

# Solution to the extension problem A

Theorem A (Trace criterion)

*Let $1 < p < \infty$ and $d > n - p$. Assume that $S$ is a $d$-thick closed subset of $\mathbb{R}^n$. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a $d$-regular system of measures. Then a Borel function $f : S \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^n)|_S$ if and only if for some $S' \subset S$ with $\mathcal{H}^d(S \setminus S') = 0$*

$$\lim_{r\to 0} \fint_{Q(x,r)} |f(x) - f(z)|\, d\mu_{k(r)}(z) = 0, \quad x \in S' \tag{2}$$

*and*

$$\mathcal{N}_{p,\lambda}[f] < \infty.$$

# Solution to the extension problem A

*Furthermore*

$$\mathcal{N}_{p,\lambda}[f] \sim \|f|W_p^1(\mathbb{R}^n)|_S\|. \qquad (3)$$

In particular for every $p > n-1$ we have an exact description of the trace space of the Sobolev space $W_p^1(\mathbb{R}^n)$ to the closure of an arbitrary domain $\Omega$!

# Extension operator

Let $S$ be a $d$-thick closed set for some $d \in [0, n]$. Let $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Let $\{Q_\varkappa\}_{\varkappa \in \mathcal{I}}$ be the Whitney decomposition of the set $\mathbb{R}^n \setminus S$. Let $\{\varphi_\varkappa\}_{\varkappa \in \mathcal{I}}$ be the corresponding partition of unity. Let $\widetilde{x}_\varkappa \in S$ be the nearest point to $Q_\varkappa$, $\varkappa \in \mathcal{I}$.

Given $f : S \to \mathbb{R}$ we set

$$
\begin{aligned}
&F(x) := \operatorname{Ext}[f](x) := \\
&= \chi_S(x) f(x) + \sum_{\substack{\varkappa \in \mathcal{I} \\ r_\varkappa \leqslant 1}} \varphi_\varkappa(x) f_\varkappa, \quad x \in \mathbb{R}^n; \\
&f_\varkappa := \fint_{Q(\widetilde{x}_\varkappa, r_\varkappa) \cap S} f(x)\, d\mu_{k(r_\varkappa)}(x), \quad \varkappa \in \mathcal{I}.
\end{aligned} \tag{4}
$$

# Solution to the Problem B

**Theorem B** *Let $S$ be a $d$-thick closed set for some $d \in [0, n]$. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Let $p \in (\max\{1, n-d\}, \infty)$. Then the operator* Ext *constructed in* (4) *acts linearly and continuously from $W_p^1(\mathbb{R}^n)|_S$ to $W_p^1(\mathbb{R}^n)$ and* $\operatorname{Tr}|_S \circ \operatorname{Ext}[f](x) = f(x)$.

# Main ingredient of the proof

Theorem Poincare-type inequality

*For $d \in [0, n]$ take a $d$-thick closed set $S$. Assume that $q \in (\max\{1, n-d\}, \infty)$ and take $F \in W_q^1(\mathbb{R}^n)$. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Then for every cube $Q = Q(x, r)$ with $x \in S$ and $r \in (0, 1]$*

$$\begin{aligned} &\fint_{Q \cap S} \left| F(y) - \fint_{Q} F(z)\, d\mathcal{L}_n(z) \right| d\mu_{k(r)}(y) \\ &\leqslant Cr \left( \fint_{Q} \sum_{|\alpha|=1} |D^{\alpha} F(t)|^q \, d\mathcal{L}_n(t) \right)^{\frac{1}{q}}, \end{aligned} \tag{5}$$

*where the constant $C > 0$ is independent of $F$ and $r$.*

# Normalized best approximations

Let $d \in [0, n]$. Let $S$ be a closed $d$-thick set in $\mathbb{R}^n$. Let $\{\mu_k\}_{k \in \mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Define for every $x \in S$ and $k \in \mathbb{N}_0$ the normalized with respect to the measure $\mu_k$ best approximation of $f$ by constant on $Q(x, 2^{-k})$

$$\mathcal{E}_{\mu_k}(f, Q(x, 2^{-k})) := \inf_{c \in \mathbb{R}} \fint_{Q(x,2^{-k}) \cap S} |f(y) - c| \, d\mu_k(y).$$

# Simplified trace functional

Assume that a function $f \in L_1^{\text{loc}}(S, \mu_k)$, $\forall k \in \mathbb{N}_0$. We define the functional

$$\mathcal{R}_{S,p}[f] := \left( \int\limits_S |f(x)|^p \, d\mu_0(x) \right)^{\frac{1}{p}} + \left( \int\limits_S \left( f^{\sharp}_{\{\mu_k\}}(x) \right)^p d\mathcal{L}_n(x) \right)^{\frac{1}{p}}$$

$$+ \left( \sum_{k=0}^{\infty} 2^{kp(1-\frac{n-d}{p})} \int\limits_{\Sigma_k} \Big( \mathcal{E}_{\mu_k}(f, Q(x, 2^{-k})) \Big)^p d\mu_k(x) \right)^{\frac{1}{p}} < \infty,$$

where $\Sigma_k := \{x \in S | \operatorname{dist}(x, \partial S) \leqslant 2^{-k}\}$, $k \in \mathbb{N}_0$

# Simplified criterion for sets with porous boundary

Theorem Simplified criterion

*Let $d \in [0, n]$ and $p \in (\max\{1, n-d\}, \infty)$. Let $S$ be a closed $d$-thick set in $\mathbb{R}^n$. Let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a $d$-regular system of measures on $S$. Assume that $\partial S$ is porous. Then $f \in W^1_p(\mathbb{R}^n)|_S$ if and only if there exists $S' \subset S$ with $\mathcal{H}^d(S \setminus S') = 0$ s.t.*

$$\lim_{r\to 0} \fint_{Q(x,r)\cap S} |f(x) - f(z)|\, d\mu_{k(r)}(z) = 0, \quad \forall x \in S'$$

*and*

$$\mathcal{R}_{S,p}[f] < \infty.$$

# Simplified criterion for sets with porous boundary

*Furthermore*

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \sim \mathcal{R}_{S,p}[f], \tag{6}$$

*and the operator* Ext *constructed in* (4) *acts linearly and continuously from* $W_p^1(\mathbb{R}^n)|_S$ *to* $W_p^1(\mathbb{R}^n)$*. Furthermore,* $\operatorname{Tr}|_S \circ \operatorname{Ext} = \operatorname{Id}$ *on* $W_p^1(\mathbb{R}^n)|_S$*.*

# Example 1

Let $S$ be Ahlfors $n$-regular set. We can take $\mu_k = \mathcal{L}_n$, $\forall k \in \mathbb{N}_0$. Then, one can show that $\mathcal{BN}_p[f] \leqslant C\mathcal{SN}_p[f]$. As a result

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \sim \|f|L_p(S)\| + \|f_S^\sharp|L_p(S)\|,$$

where

$$f_S^\sharp(x) := \sup_{r\in(0,1)} \frac{1}{r} \fint_{Q(x,r)\cap S} \left| f(y) - \fint_{Q(x,r)} f(z)\, d\mathcal{L}_n(z) \right| d\mathcal{L}_n(y).$$

This result coincides with corresponding result obtained by P. Shvartsman in 2006.

# Example 2

Let $S$ be an Ahlfors $d$-regular set with $d \in (0, n)$. Then, $S$ is porous and $\mu_k = \mathcal{H}^d$, $\forall k \in \mathbb{N}_0$. One can show that

$$\mathcal{BN}_p[f] \approx \|f|L_p(S, \mathcal{H}^d)\| + \left( \sum_{j=0}^{\infty} 2^{jp(1-\frac{n-d}{p})} \int_S \widetilde{\mathcal{E}}(f, Q(x, \frac{1}{2^j}))\, d\mathcal{H}^d(x) \right)^{\frac{1}{p}}$$

where

$$\widetilde{\mathcal{E}}(f, Q(x, 2^{-j})) = \fint_{Q(x,2^{-j})} \left| f(y) - \fint_{Q(x,2^{-j})} f(z)\, d\mathcal{H}^d(z) \right| d\mathcal{H}^d(y).$$

It is clear that $\mathcal{SN}_p[f] = 0$. As a result,

$$\|f|W_p^1(\mathbb{R}^n)|_S\| \approx \mathcal{BN}_p[f].$$

This result coincides with corresponding result obtained by L. Ihnatseva in 2011.

# Thank you for your attention

THANK YOU FOR YOUR ATTENTION !

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