Asymptotic stability of stationary solutions to the Navier-Stokes equation in Besov spaces

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Non-stationary Navier-Stokes equation

Incompressible Navier-Stokes equation:

$$\text{(N-S)} \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \pi = f, & \text{in} \quad (0, \infty) \times \mathbb{R}^n, \ (n \geq 3) \\ \operatorname{div} u = 0, \\ u(0) = a. \end{cases}$$

Here
∘Unknown: $u = (u_1, \cdots, u_n)$: velocity, π : pressure.
∘Given data: $f = (f_1, \cdots, f_n)$: external force, $a = (a_1, \cdots, a_n)$: initial data.
Remark that div $u (= \nabla \cdot u) = 0 \Rightarrow$

$$(u \cdot \nabla) u := \left(\sum_{j=1}^n u_j \partial_j u_1, \ \cdots \ \sum_{j=1}^n u_j \partial_j u_n \right) = \nabla (u \otimes u).$$

Stationary Navier-Stokes equation

$u(t,x) = U(x) \Rightarrow$

$$\text{(S)} \quad \begin{cases} -\Delta U + \nabla(U \otimes U) + \nabla \pi = f \\ \text{div } U = 0. \end{cases}$$

$\circ \mathbb{P} := (\delta_{i,j} I + R_i R_j)_{1 \le i,j \le n}$: Helmholtz projection fulfills

$$\begin{cases} \text{div} g = 0 \Rightarrow \mathbb{P}(g) = g \\ \mathbb{P}(\nabla \pi) = 0. \end{cases}$$

$\circ (S) \xrightarrow{\mathbb{P}}$

$$\begin{cases} -\Delta U + \mathbb{P}\nabla(U \otimes U) = \mathbb{P} f \\ \text{div } U = 0. \end{cases}$$

$\Longleftrightarrow$

$$(S)^* : \quad U = (-\Delta)^{-1} \mathbb{P} f - (-\Delta)^{-1} \mathbb{P} \nabla (U \otimes U).$$

Existence of U. $(S)^*$: $U = (-\Delta)^{-1}\mathbb{P}f - (-\Delta)^{-1}\mathbb{P}\nabla(U \otimes U)$

Let

$$n \geq 3, \quad \frac{n}{2} < p < n \quad \text{and} \quad s(p) := -1 + n/p \in (0,1).$$

$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \left\{ 2^{js} \|\varphi_j(D) f\|_{L^p} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q}, \ \varphi_j(D) f := \mathcal{F}^{-1}[\varphi(\cdot/2^j)\hat{f}].$$

Theorem 1

(i): If $\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} \ll 1$, then $\exists! U \in \dot{B}^{s(p)}_{p,\infty}$ solving $(S)^*$ in $\dot{B}^{s(p)}_{p,\infty}$ and

$$\|U\|_{\dot{B}^{s(p)}_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} \ll 1.$$

(ii): Let $s \in (0,1)$. If $\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} \ll 1$ and $f \in \dot{B}^{s-2}_{p,\infty}$, then the solution U in (i) also satisfies

$$\|U\|_{\dot{B}^s_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s-2}_{p,\infty}}.$$

Aim: Asymptotic stability of U

When $\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} \ll 1$ and $\|a - U\|_{\dot{B}^{s(p)}_{p,\infty}} \ll 1$, there exists the solution u of

$$\mathbb{P}\text{(N.S.)} \quad \begin{cases} \partial_t u - \Delta u + \mathbb{P}\nabla(u \otimes u) = \mathbb{P}f, & \text{in} \quad (0,\infty) \times \mathbb{R}^n, \ (n \geq 3) \\ \text{div } u = 0, \\ u(0) = a(= U + (a - U)) \end{cases}$$

satisfying

$$\|u(t) - U\|_{\dot{B}^{s(p)\pm\tau}_{p,\infty}} \to 0 \quad \text{as } t \to \infty$$

for some $\tau > 0$.

Main result: High frequency decay: $\|u(t) - U\|_{\dot{B}^{s(p)+\tau_H}_{p,1}} \to 0$

Theorem 2

(i) Let $\tau_H \in (0, 1 - s(p))$. If $\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} + \|a - U\|_{\dot{B}^{s(p)}_{p,\infty}} \ll 1$, then $^\exists u \in BC((0,\infty); \dot{B}^{s(p)}_{p,\infty})$ so that $u - U \in BC((0,\infty); \dot{B}^{s(p)}_{p,\infty} \cap \dot{B}^{s(p)+\tau_H}_{p,1})$ solving

$$\partial_t u - \Delta u + \mathbb{P}\nabla(u \otimes u) = \mathbb{P}f \quad \text{in} \quad \dot{B}^{s(p)-2}_{p,\infty}$$

with the initial condition $u(0) = a$ in the sense that for $\alpha \in [0, 2]$

$$\|u(t) - a\|_{\dot{B}^{s(p)-\alpha}_{p,\infty}} \lesssim t^{\alpha/2}\|a - U\|_{\dot{B}^{s(p)}_{p,\infty}}.$$

Moreover, $\|u(t) - U\|_{\dot{B}^{s(p)}_{p,\infty}} \lesssim \|a - U\|_{\dot{B}^{s(p)}_{p,\infty}}$ and

$$\|u(t) - U\|_{\dot{B}^{s(p)+\tau_H}_{p,1}} \lesssim t^{-\tau_H/2}\|a - U\|_{\dot{B}^{s(p)}_{p,\infty}}. \tag{1}$$

Main result: Low frequency decay: $\|u(t) - U\|_{\dot{B}^{s(p)-\tau_L}_{p,\infty}} \to 0$

Theorem 2

(ii) Let $s \in (0, s(p))$. If $\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} + \|a - U\|_{\dot{B}^{s(p)}_{p,\infty}} \ll 1,\ f \in \dot{B}^{s-2}_{p,\infty}$ and $a - U \in \dot{B}^{s}_{p,\infty}$, then the solution u constructed in (i) satisfies $u \in BC((0,\infty); \dot{B}^{s(p)}_{p,\infty} \cap \dot{B}^{s}_{p,\infty})$, and for $\tau_L \in (0, s(p) - s]$

$$\|u(t) - U\|_{\dot{B}^{s(p)-\tau_L}_{p,\infty}} \lesssim t^{-\gamma/2}\|a - U\|_{\dot{B}^{s}_{p,\infty}}, \tag{2}$$

where $\gamma := s(p) - \tau_L - s > 0$.

Remark: L^p decay

1. (1) $\Rightarrow \|u(t)-U\|_{L^{n/(1-\tau_H)}} \lesssim \|u(t)-U\|_{\dot{B}^{s(p)+\tau_H}_{p,1}} \lesssim t^{-\tau_H/2}$
2. (2) $\Rightarrow \|u(t)-U\|_{L^{n/(1+\tau_L),\infty}} \lesssim \|u(t)-U\|_{\dot{B}^{s(p)-\tau_L}_{p,\infty}} \lesssim t^{-\gamma/2}$

where

$$p < \frac{np}{n-sp} \leq \frac{n}{1+\tau_L} < n < \frac{n}{1-\tau_H} < \frac{np}{n-p}.$$

Remark

1. Kozono-Yamazaki ('95): High frequency decay in Besov-Morrey spaces.
2. Bjorland, Brandolese, Iftimie and Schonbek ('11): Low frequency decay in weak L^p spaces.

Construction of U: $U = (-\Delta)^{-1}\mathbb{P}f - (-\Delta)^{-1}\mathbb{P}\nabla(U \otimes U)$.

$$U_0 := (-\Delta)^{-1}\mathbb{P}f \quad \text{and} \quad U_{m+1} := U_0 - (-\Delta)^{-1}\mathbb{P}\nabla(U_m \otimes U_m).$$

Lemma 1

Let $1 < p < \infty,\ s < n/p$ and $\ell := np/(n - sp)$.
(i) $0 < s < n/p \Rightarrow \dot{B}^s_{p,\infty} \hookrightarrow L^{\ell,\infty}$.
(ii) $s < 0 \Rightarrow L^{\ell,\infty} \hookrightarrow \dot{B}^s_{p,\infty}$.

Lemma 2

If $0 < s < 1$, then

$$\|fg\|_{\dot{B}^{s-1}_{p,\infty}} \lesssim \|f\|_{L^{n,\infty}}\|g\|_{\dot{B}^s_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s(p)}_{p,\infty}}\|g\|_{\dot{B}^s_{p,\infty}}, \ (s(p) = -1 + n/p).$$

Lemma 1 + Lemma 2 $\Rightarrow \{U_m\}_{m=0}^{\infty}$: Cauchy seq. in $\dot{B}^{s(p)}_{p,\infty} \cap \dot{B}^s_{p,\infty}$.

$$-\Delta U + \mathbb{P}\nabla(U \otimes U) = \mathbb{P}f \text{ in } \dot{B}^{s(p)-2}_{p,\infty}.$$

Construction of non-stationary solution u

Recall: $n \geq 3$ and $n/2 < p < n$. Let $\|U\|_{\dot{B}^{s(p)}_{p,\infty}} \ll 1$ and

$$\begin{cases} \mathcal{B}[w] := \mathbb{P}\nabla(w \otimes U + U \otimes w) \\ \mathcal{A}[w] := -\Delta w + \mathcal{B}[w] \end{cases}$$

If U is a stationary solution and w satisfies

$$(*) \begin{cases} \partial_t w + \mathcal{A}[w] + \mathbb{P}\nabla(w \otimes w) = 0 \\ \text{div } w = 0 \\ w(0) = b := a - U, \end{cases}$$

then $u(t) := w(t) + U$ satisfies

$$\mathbb{P}\text{(N-S)} \begin{cases} \partial_t u - \Delta u + \mathbb{P}\nabla(u \otimes u) = \mathbb{P}f \\ \text{div } u = 0, \\ u(0) = a \end{cases}$$

and $\|w(t)\| \to 0, \ (t \to \infty) \iff \|u(t) - U\| \to 0, \ (t \to \infty)$.

Resolvent estimates

Let $\|U\|_{\dot{B}^{s(p)}_{p,\infty}} \ll 1$ and $\lambda \in S_\omega := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg(\lambda)| > \omega\},\ (0 < \omega < \pi/2)$.
The resolvent operator

$$R_{\mathcal{A}}(\lambda) := (\lambda - \mathcal{A})^{-1} = (1 - R_{-\Delta}(\lambda)\mathcal{B})^{-1} R_{-\Delta}(\lambda)$$

has the smoothing property:

Proposition 1

Let $-2 < s < 1$ and $0 \le \tau \le 2$ with $s + \tau < 1$. Then

$$\|R_{\mathcal{A}}(\lambda) f\|_{\dot{B}^{s+\tau}_{p,\infty}} \lesssim |\lambda|^{-(2-\tau)/2} \|f\|_{\dot{B}^{s}_{p,\infty}}.$$

This can be proved from an argument of Kozono-Yamazaki (1995):

$$R_{\mathcal{A}}(\lambda) = R_{-\Delta}(\lambda) + R_{-\Delta}(\lambda)\mathcal{B}R_{\mathcal{A}}(\lambda)$$
$$R_{-\Delta}(\lambda) := (\lambda + \Delta)^{-1}.$$

Resolvent estimates for $-\Delta$

We define Fourier multiplier operator $m(D)$ on $\dot{B}^{s(p)}_{p,\infty}$ by

$$m(D)f := \sum_{j\in\mathbb{Z}} \mathscr{F}^{-1}\left[m\varphi(\cdot/2^j)\hat{f}\right].$$

Since $m_\lambda(\xi) := (\lambda - |\xi|^2)^{-1}$ satisfies

$$|\partial^\alpha m_\lambda(\xi)| \lesssim \min\left(|\lambda|^{-1}, |\xi|^{-2}\right)|\xi|^{-|\alpha|} \text{ for } {}^\forall \xi \neq 0,$$

then from Mikhlin's theorem,

$$\begin{aligned} \|R_{-\Delta}(\lambda)f\|_{\dot{B}^s_{p,\infty}} &= \|m_\lambda(D)f\|_{\dot{B}^s_{p,\infty}} \lesssim |\lambda|^{-1}\|f\|_{\dot{B}^s_{p,\infty}} \\ \|R_{-\Delta}(\lambda)f\|_{\dot{B}^{s+2}_{p,\infty}} &= \|m_\lambda(D)f\|_{\dot{B}^{s+2}_{p,\infty}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}}, \end{aligned}$$

hence

$$\|R_{-\Delta}(\lambda)f\|_{\dot{B}^{s+\tau}_{p,\infty}} \lesssim |\lambda|^{-(2-\tau)/2}\|f\|_{\dot{B}^s_{p,\infty}}.$$

Smoothing estimates for the semigroup $e^{-t\mathcal{A}}$

$$e^{-t\mathcal{A}}f := \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}R_{\mathcal{A}}(\lambda)f\,d\lambda$$

with the contour $\Gamma \subset S_\omega$, which is oriented counterclockwise and connects $e^{-i\theta}\infty$ and $e^{i\theta}\infty$ for some $0 < \omega < \theta < \pi/2$.

Proposition 2

Let $-2 < s < 1$.

(i)

$$\|e^{-t\mathcal{A}}f\|_{\dot{B}^s_{p,\infty}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}}.$$

(ii)

$$\|e^{-t\mathcal{A}}f\|_{\dot{B}^{s+\tau}_{p,1}} \lesssim t^{-\tau/2}\|f\|_{\dot{B}^s_{p,\infty}} \quad \text{if } 0 < \tau < 1 - s.$$

(iii) If $0 < s < 1$ and $0 \le \tau < s$, then

$$\|e^{-t\mathcal{A}}f - f\|_{\dot{B}^{s-\tau-2}_{p,\infty}} \lesssim t^{\tau/2}\|f\|_{\dot{B}^s_{p,\infty}}.$$

Construction of w

$\|f\|_{\dot{B}^{s(p)-2}_{p,\infty}} \ll 1$ & $U \in \dot{B}^{s(p)}_{p,\infty}$ is the stationary solution in Theorem 1.

$$\mathcal{B}[w] := \mathbb{P}\nabla(w \otimes U + U \otimes w), \quad \mathcal{A}[w] := -\Delta w + \mathcal{B}[w]$$

$$(*) \begin{cases} \partial_t w + \mathcal{A}[w] + \mathbb{P}\nabla(w \otimes w) = 0 \\ \text{div } w = 0 \\ w(0) = b := a - U, \end{cases}$$

with $\|w(t)\| \to 0$ as $t \to \infty$.

◦ Integral form:

$$w(t) = e^{-t\mathcal{A}} b - B(w, w)(t),$$

where

$$B(g, h)(t) := \int_0^t e^{-(t-\sigma)\mathcal{A}} \mathbb{P}\nabla(g \otimes h)(\sigma) d\sigma.$$

Meyer type critical estimate

Proposition 3

For $s \in (0, 1)$,

$$\left\| \int_{t_0}^{t} e^{-(t-\sigma)\mathcal{A}} \mathbb{P} g(\sigma) d\sigma \right\|_{\dot{B}^s_{p,\infty}} \lesssim \sup_{t_0 \le \sigma \le t} \|g(\sigma)\|_{\dot{B}^{s-2}_{p,\infty}},$$

where $-\infty \le t_0 < t < \infty$.

Remark

From the smoothing estimate,

$$\|e^{-(t-\sigma)\mathcal{A}} \mathbb{P} g(\sigma)\|_{\dot{B}^s_{p,\infty}} \lesssim (t-\sigma)^{-1} \|g(\sigma)\|_{\dot{B}^{s-2}_{p,\infty}}$$

Meyer type critical estimate

Remark

Proposition 3 is related to L^∞*−maximal regularity:*

If $u(t) := \int_0^t e^{-(t-\sigma)\mathcal{A}} f(\sigma)d\sigma$ *solves*

$$\partial_t u + \mathcal{A}u = f,$$

then we have from Proposition 3 eliminating $\mathbb{P}$

$$\|\partial_t u\|_{L^\infty\left((0,\infty);\dot{B}^{s(p)-2}_{p,\infty}\right)} + \|\mathcal{A}u\|_{L^\infty\left((0,\infty);\dot{B}^{s(p)-2}_{p,\infty}\right)} \lesssim \|f\|_{L^\infty\left((0,\infty);\dot{B}^{s(p)-2}_{p,\infty}\right)}.$$

A proof of Proposition 3

Let $\tilde{g}(\sigma) := g(t-\sigma)\chi_{(0,t-t_0)}(\sigma)$ and $0 < \varepsilon \ll 1$.

$$\left\| \int_{t_0}^{t} e^{-(t-\sigma)\mathcal{A}} \mathbb{P} g(\sigma) d\sigma \right\|_{\dot{B}^s_{p,\infty}} = \left\| \int_0^\infty e^{-\sigma\mathcal{A}} \mathbb{P} \tilde{g}(\sigma) d\sigma \right\|_{\dot{B}^s_{p,\infty}} \approx \| \cdots \|_{(\dot{B}^{s-\varepsilon}_{p,\infty}, \dot{B}^{s+\varepsilon}_{p,\infty})_{1/2,\infty}}$$

$$\leq \sup_{\lambda>0} \lambda^{-1/2} \left(\left\| \int_0^{\lambda_*} \cdots d\sigma \right\|_{\dot{B}^{s-\varepsilon}_{p,\infty}} + \lambda \left\| \int_{\lambda_*}^\infty \cdots d\sigma \right\|_{\dot{B}^{s+\varepsilon}_{p,\infty}} \right).$$

Smoothing estimate tells us

$$\left\| \int_0^{\lambda_*} \cdots d\sigma \right\|_{\dot{B}^{s-\varepsilon}_{p,\infty}} \lesssim {\lambda_*}^{\varepsilon/2} \sup_{\sigma>0} \|g(\sigma)\|_{\dot{B}^{s-2}_{p,\infty}}$$

$$\left\| \int_{\lambda_*}^\infty \cdots d\sigma \right\|_{\dot{B}^{s+\varepsilon}_{p,\infty}} \lesssim {\lambda_*}^{-\varepsilon/2} \sup_{\sigma>0} \|g(\sigma)\|_{\dot{B}^{s-2}_{p,\infty}}$$

Taking $\lambda_* = \lambda^{1/\varepsilon}$ gives the desired inequality.

Thank you for your attention !

Dziękuję za uwagę !