PDE models for chemotaxis in supercritical function spaces

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Chemotaxis: Movement biological cells in response of chemical gradient.

Interest: Ability to initiate spacial pattering caused by chemotactical migration, producing or consuming chemicals. Patterns (dots, strips etc.) on skins, also on a macroscopic scale (spots of a leopard).

Stars: 1. Slime mo(u)ld Dictyostelium discoideum, amoebae. Observed since the 1950s, fundamental discovery early 1990s (Biol. Dept. Havard Univ., in Nature): Beautiful patterns in Petri dishes, $\mathbb{R}^2$-matter. Circular, diameter $\sim 10^{-2}$ mm, $5 \cdot 10^4$ per cm^2 needed, 10^5 cells behave like a single organism, produces auto-attractant cAMP (monophosphate) which initiates an aggregation process.

2. Aerobic bacteria Bacillus subtilis, since 2005. Observed in a droplet in $\mathbb{R}^3$, influenced by buoyancy [vertical upward force of a fluid on a floating or immersed body, which is equal to the weight of the fluid displaced by the body, Archimedes, heureka]. Like a cigar, $5 \cdot 10^{-3}$ mm $\times 0.8 \cdot 10^{-3}$ mm. Consumes oxygen molecules, 10^6 per second. Needed cell density 10^9 per cm^3: Oxygentaxis.

Models. Observed since 1930s, first mathematical models in 1950s, break-through *Keller-Segel*, 1970–71. Dimensions 2 and 3. Here: All in $\mathbb{R}^n$.

Parabolic-parabolic model (for Dd [Dictyostelium discoideum])

$$\begin{aligned} \partial_t u - \Delta u + \mathrm{div}\,(u\nabla v) &= 0, && x \in \mathbb{R}^n,\ 0 < t < T, \\ \partial_t v - \Delta v + \alpha v &= u, && x \in \mathbb{R}^n,\ 0 < t < T, \\ u(\cdot, 0) &= u_0, && x \in \mathbb{R}^n, \\ v(\cdot, 0) &= v_0, && x \in \mathbb{R}^n, \end{aligned}$$

$\alpha \geq 0$, the so-called damping constant. $\alpha > 0$ is of biological relevance, positive chemotaxis. $u(x,t)$: cell-density, $v(x,t)$ concentration of chemical.

$$\mathrm{div}\, w = \sum_{j=1}^{n} \partial_j w^j, \qquad \nabla v = (\partial_1 v, \ldots, \partial_n v).$$

Chemotaxis Navier–Stokes equations: (for Bacillus subtilis)

$$
\begin{aligned}
\partial_t u - \Delta u + \operatorname{div}(u\nabla v) + w\cdot\nabla u &= 0 && \text{in } \mathbb{R}^n\times(0,T),\\
\partial_t v - \Delta v + uv + w\cdot\nabla v &= 0 && \text{in } \mathbb{R}^n\times(0,T),\\
\partial_t w - \Delta w + (w,\nabla)w - u\nabla\Phi + \nabla P &= 0 && \text{in } \mathbb{R}^n\times(0,T),\\
\operatorname{div} w &= 0 && \text{in } \mathbb{R}^n\times(0,T),\\
u(\cdot,0) &= u_0 && \text{in } \mathbb{R}^n,\\
v(\cdot,0) &= v_0 && \text{in } \mathbb{R}^n,\\
w(\cdot,0) &= w_0 && \text{in } \mathbb{R}^n.
\end{aligned}
$$

$u(x,t)$: cell-density, $v(x,t)$ concentration of chemical, $w(x,t) = (w^1(x,t),\dots,w^n(x,t))$ velocity of the fluid, $P(x,t)$ unknown scalar pressure, Φ given gravitational potential (buoyancy).

$$
w\cdot\nabla u = \sum_{j=1}^{n} w^j\,\partial_j u.
$$

$$
[(w,\nabla)w]^k = \sum_{j=1}^{n} w^j\,\partial_j w^k, \qquad k = 1,\dots,n,
$$

$$
\operatorname{div} w = \sum_{j=1}^{n} \partial_j w^j, \qquad \nabla P = (\partial_1 P,\dots,\partial_n P).
$$

All in $\mathbb{R}^n$. $A^s_{p,q}$ either $B^s_{p,q}$ or $F^s_{p,q}$ as usual, $s \in \mathbb{R}$, $0 < p, q \le \infty$ ($p < \infty$ for F-spaces). $\varphi = \{\varphi\}_{j=0}^{\infty}$ the usual dyadic resolution of unity.

$$\|f \,|B^s_{p,q}\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \big\| (\varphi_j \widehat{f})^\vee \,|L_p\big\|^q \Big)^{1/q},$$

$$\|f \,|F^s_{p,q}\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \big| (\varphi_j \widehat{f})^\vee(\cdot)\big|^q \Big)^{1/q} \Big| L_p \Big\|.$$

Special cases. $L_p = F^0_{p,2}, 1 < p < \infty$. As usual,

$$\|f \,|L_p\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p}, \qquad 0 < p \le \infty,$$

natural modification if $p = \infty$. Classical Sobolev spaces $W^k_p = F^k_{p,2}$, $1 < p < \infty$, $k \in \mathbb{N}_0$,

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha| \le k} \big\| D^\alpha f \,|L_p(\mathbb{R}^n)\big\|^p \Big)^{1/p}.$$

(Fractional) Sobolev spaces $H^s_p = F^s_{p,2}$, $s \in \mathbb{R}$, $1 < p < \infty$,

$$\|f \,|H^s_p\| = \Big\| \big((1 + |\xi|^2)^{s/2} \widehat{f} \big)^\vee \Big| L_p \Big\|.$$

$H^s_p = W^k_p$ if $s = k \in \mathbb{N}_0$.

Hölder-Zygmund spaces $\mathcal{C}^s = B^s_{\infty,\infty}$, $s \in \mathbb{R}$.

$$\big(\Delta^1_h f\big)(x) = f(x+h) - f(x), \qquad \big(\Delta^{m+1}_h f\big)(x) = \Delta^1_h\big(\Delta^m_h f\big)(x),$$

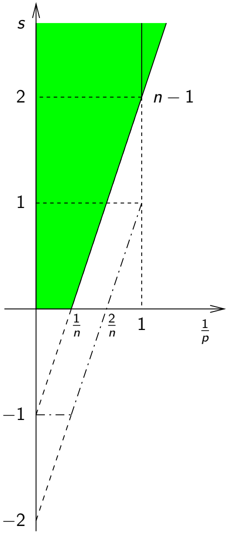
$m \in \mathbb{N}$, iterated differences in $\mathbb{R}^n$. If $0 < s < m \in \mathbb{N}$ then

$$\|f\,|\mathcal{C}^s\|_m = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-s}\big|\Delta^m_h f(x)\big|$$

second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$.
Classical Besov spaces $0 < s < m \in \mathbb{N}$ and $1 \le p \le \infty$, $0 < q \le \infty$ then (equivalent norms)

$$\|f\,|B^s_{p,q}\|_m = \|f\,|L_p\| + \Big(\int_{|h|\le 1} |h|^{-sq}\big\|\Delta^m_h f\,|L_p\big\|^q \,\frac{\mathrm{d}h}{|h|^n}\Big)^{1/q}$$

Other $B^s_{p,q}$ by lifting.


s
2
n − 1
1
1/n
2/n
1
1/p
−1
−2

Homogeneous spaces $\dot{A}^s_{p,q}$ or $\overset{*}{A}{}^s_{p,q}$. Homogeneity:

$$\|f(\lambda \cdot) \,|\overset{*}{A}{}^s_{p,q}\| = \lambda^{s-\frac{n}{p}} \,\|f\,|\overset{*}{A}{}^s_{p,q}\|, \qquad \lambda > 0.$$

Critical spaces:

$$s - \frac{n}{p} = \begin{cases} -1 & \text{for Navier-Stokes equations,} \\ -2 & \text{for Keller-Segel equations.} \end{cases}$$

Homogeneous equations $\alpha = 0$, or, simpler parabolic-elliptic model with $\alpha = 0$.

Proposition 1. Let $\lambda > 0$ and $u_0 = u_0(x)$, $x \in \mathbb{R}^n$. Let $u^\lambda = u^\lambda(x,t)$, $v^\lambda = v^\lambda(x,t)$, $x \in \mathbb{R}^n$, $0 \le t < T$ be a solution of

$$\begin{aligned} \partial_t u^\lambda - \Delta u^\lambda + \sum_{j=1}^n \partial_j u^\lambda \cdot \partial_j v^\lambda - (u^\lambda)^2 &= 0, && x \in \mathbb{R}^n,\ 0 < t < T, \\ -\Delta v^\lambda &= u^\lambda, && x \in \mathbb{R}^n,\ 0 < t < T, \\ u^\lambda(x,0) &= \lambda^{-2} u_0(\lambda^{-1}x), && x \in \mathbb{R}^n. \end{aligned}$$

Then

$$\begin{aligned} u_\lambda(x,t) &= \lambda^2 u^\lambda(\lambda x, \lambda^2 t), && x \in \mathbb{R}^n,\ 0 \le t < \lambda^{-2}T, \\ v_\lambda(x,t) &= v^\lambda(\lambda x, \lambda^2 t), && x \in \mathbb{R}^n,\ 0 \le t < \lambda^{-2}T, \end{aligned}$$

is a solution of

$$\begin{aligned} \partial_t u_\lambda - \Delta u_\lambda + \sum_{j=1}^n \partial_j u_\lambda \cdot \partial_j v_\lambda - u_\lambda^2 &= 0, && x \in \mathbb{R}^n,\ 0 < t < \lambda^{-2}T, \\ -\Delta v_\lambda &= u_\lambda, && x \in \mathbb{R}^n,\ 0 < t < \lambda^{-2}T, \\ u_\lambda(x,0) &= u_0(x), && x \in \mathbb{R}^n. \end{aligned}$$

Discussion 2. Assumption: There is an $\delta > 0$ such that above model case with $\lambda = 1$ has for any $\| u_0 \,|\overset{*}{A}{}^s_{p,q}\| \le \delta$ a solution in $\mathbb{R}^n \times (0, T)$ for some $T > 0$. This requires applied to $u^\lambda(x, 0)$

$$\| u_0 \,|\overset{*}{A}{}^s_{p,q}\| \le \delta\, \lambda^{2+s-\frac{n}{p}}, \qquad \lambda > 0.$$

Then u_λ solution in $\mathbb{R}^n \times (0, \lambda^{-2} T)$.

Critical, $s - \frac{n}{p} = -2$: $\lambda \to 0$, then same δ, $\lambda^{-2} T \to \infty$, (global solutions for small initial data),

supercritical, $s - \frac{n}{p} > -2$. $\lambda \to \infty$ then large u_0 in small time $\lambda^{-2} T \to 0$,

subcritical, $s - \frac{n}{p} < -2$. $\lambda \to 0$ then large u_0 in large time $\lambda^{-2} T \to \infty$.

Guide also for the above classical Keller-Segel equation in inhomogeneous spaces.

Subcritical case: Unlikely. Critical: Some papers (Navier-Stokes people, Kozono, Lemarié-Rieusset, others.) Here: Supercritical case. For bounded domains Ω in $\mathbb{R}^n$ situation different, papers in classical spaces of type C^k.

Gauss-Weierstrass semi-group:

$$W_t w(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} w(y)\,\mathrm{d}y, \qquad t > 0,$$

$$\widehat{W_t w}(\xi) = e^{-t|\xi|^2} \widehat{w}(\xi), \qquad \xi \in \mathbb{R}^n, \quad t > 0.$$

Linear heat equation:

$$\begin{aligned} \partial_t W(x,t) - \Delta W(x,t) &= f(x,t), & x \in \mathbb{R}^n, \quad t > 0, \\ W(x,0) &= w(x), & x \in \mathbb{R}^n. \end{aligned}$$

Solution: Duhamel formula,

$$\begin{aligned} W(x,t) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} w(y)\,\mathrm{d}y + \frac{1}{(4\pi)^{n/2}} \int_0^t \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}}{(t-\tau)^{n/2}} f(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &= W_t w(x) + \Big(\int_0^t W_{t-\tau} f_\tau \,\mathrm{d}\tau \Big)(x), \end{aligned}$$

where $f_\tau(y) = f(y,\tau)$. Reformulation on the Fourier side

$$\widehat{W}(\xi, t) = e^{-t|\xi|^2} \widehat{w}(\xi) + \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{f_\tau}(\xi)\,\mathrm{d}\tau.$$

Proposition 3. Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $d \ge 0$ and $0 < t \le 1$. Then

$$t^{d/2} \, \| W_t w \, | A^{s+d}_{p,q} \| \le c \, \| w \, | A^s_{p,q} \|.$$

Remark 4. Crucial instrument, also for Navier-Stokes equations. Price $t^{-d/2}$ to pay for better smoothness $s + d$.

Recall: Keller-Segel equation:

$$\begin{aligned} \partial_t u - \Delta u + \mathrm{div}\,(u\nabla v) &= 0, && x \in \mathbb{R}^n,\ 0 < t < T, \\ \partial_t v - \Delta v + \alpha v &= u, && x \in \mathbb{R}^n,\ 0 < t < T, \\ u(\cdot, 0) &= u_0, && x \in \mathbb{R}^n, \\ v(\cdot, 0) &= v_0, && x \in \mathbb{R}^n, \end{aligned}$$

$\alpha \geq 0$, where $u = u(x,t)$, $v = v(x,t)$ and $u_0 = u_0(x)$, $v_0 = v_0(x)$. One has to insert

$$v(x,t) = e^{-\alpha t}\Big(\int_0^t e^{\alpha\tau} W_{t-\tau} u(\cdot,\tau)\,\mathrm{d}\tau\Big)(x) + e^{-\alpha t} W_t v_0(x)$$

in the first equation. Then one has the nonlinearity

$$Pu = \mathrm{div}\,(u\nabla v) = u \cdot \Delta v + \sum_{j=1}^{n} \partial_j u \cdot \partial_j v.$$

Method: Convert Keller-Segel equation in related fixed point problems (Banach's contraction theorem) based on the Duhamel formula.

$$T_{u_0} u(x,t) = W_t u_0(x) - \Big(\int_0^t W_{t-\tau} Pu(\cdot,\tau)\, \mathrm{d}\tau \Big)(x)$$

with

$$f_\tau = -Pu = -\mathrm{div}\,(u \nabla v).$$

If u is a fixed point, hence $T_{u_0} u = u$, then u is a solution with initial data u_0. Afterwards solution for v with initial data v_0 as above. Pu quadratic nonlinearity, considered in $L_\infty\big((0,T), a/2, A^s_{p,q}\big)$, normed by

$$\| u \,| L_\infty\big((0,T), a/2, A^s_{p,q}\big) \| = \sup_{0<t<T} t^{a/2} \| u(\cdot,t) \,| A^s_{p,q} \|.$$

Instruments: (i) Multiplication algebras:

$$A^s_{p,q} \cdot A^s_{p,q} \hookrightarrow A^s_{p,q}, \qquad s > n/p.$$

(ii) Hölder inequalities:

$$2 < r < \infty, \quad s > 0, \quad \frac{1}{p} = \frac{1}{r} + \frac{s}{n}, \quad 0 < q \le \infty.$$

$$A^s_{p,q} \cdot A^s_{p,q} \hookrightarrow A^s_{p_r,q}, \qquad \frac{1}{p_r} = \frac{1}{p} + \frac{1}{r},$$

$0 < q \le r$ in the case of B-spaces.
(iii) Pointwise multipliers:

$$A^\sigma_{\varrho,\delta} \cdot A^s_{p,q} \hookrightarrow A^s_{p,q}, \quad 1 \le p < \infty, \quad 0 < s < n/p, \quad \sigma > \max\big(s, n/\varrho\big).$$

(iv) Duality, embeddings, lifts.
(v) Proposition 3.

Proposition 5. Let $2 \le n \in \mathbb{N}$,

$$1 \le p, q \le \infty, \qquad s > \Big(\frac{n}{p} - 1\Big)_+ \quad \text{and} \quad 0 < a < 2,$$

Figure. Let $0 < \varepsilon < a/2$ and $v_0 \in A^{s+2-a}_{p,q}$. Then

$$\begin{aligned} &\| Pu(\cdot, t) \, | A^{s-1-2\varepsilon}_{p,q} \| \\ &\le c_\varepsilon \, t^{\varepsilon - \frac{a}{2}} \, \| u(\cdot, t) \, | A^s_{p,q} \| \cdot \Big(\| v_0 \, | A^{s+2-a}_{p,q} \| + \sup_{0<\tau<t} \tau^{\frac{a}{2}} \| u(\cdot, \tau) \, | A^s_{p,q} \| \Big) \end{aligned}$$

for some $c_\varepsilon > 0$, all $0 < t \le 1$ and all $u(\cdot, t) \in A^s_{p,q}$.

Remark 6. Pointwise multiplier theorem with time memory.

Theorem 6. Let $2 \le n \in \mathbb{N}$. Let $1 \le p, q \le \infty$ ($p < \infty$ for F-spaces), $s > \left(\frac{n}{p} - 1\right)_+$.

Let

$$0 < g \le 1 \quad \text{and} \quad a = 1 - \varkappa g \quad \text{with} \quad 0 < \varkappa < 1.$$

Let

$$u_0 \in A^{s-1+g}_{p,q} \qquad \text{and} \qquad v_0 \in A^{s+1+\varkappa g}_{p,q}.$$

Then there is a number T, $T > 0$, such that the Keller-Segel equation with $\alpha \ge 0$ has a unique mild solution

$$u \in L_\infty\big((0,T), a/2, A^s_{p,q}\big).$$

Furthermore, $u \in C^\infty\big(\mathbb{R}^n \times (0,T)\big)$. If, in addition, $p < \infty$, $q < \infty$,

$$0 < g \le 1 \quad \text{and} \quad a = 1 - \varkappa g \quad \text{with} \quad 1/2 < \varkappa < 1$$

then the above solution is strong, that means $u \in C\big([0,T), A^{s-1+g}_{p,q}\big)$.

Remark 7. Similarly v strong solution,

$$v \in C\big([0,T), A^{s+1+\varkappa g}_{p,q}\big).$$

Stability: For any $\varepsilon > 0$ there are $T > 0$ and $\delta > 0$ such that

$$\| u^1(\cdot, t) - u^2(\cdot, t) \, | X \| + \| v^1(\cdot, t) - v^2(\cdot, t) \, | Y \| \le \varepsilon$$

whenever

$$0 \le t \le T \quad \text{and} \quad \| u_0^1 - u_0^2 \, | X \| + \| v_0^1 - v_0^2 \, | Y \| \le \delta$$

with $X = A^{s-1+g}_{p,q}$ and $Y = A^{s+1+\varkappa g}_{p,q}$ as in Theorem 6.

There is a positive answer, including interplay of δ, t, ε.

Well-posed: mild + unique + strong +stable.

Covered by Theorem 6 and stability.

Simplest case: If

$$u_0 \in S(\mathbb{R}^n) \quad \text{and} \quad v_0 \in S(\mathbb{R}^n)$$

then

$$u(\cdot, t) \in S(\mathbb{R}^n) \quad \text{and} \quad v(\cdot, t) \in S(\mathbb{R}^n), \qquad 0 < t < T.$$

Proof: Let $w_\gamma(x) = (1 + |x|^2)^{\gamma/2}$, $\gamma \in \mathbb{R}$ and

$$A^s_{p,q}(\gamma) = \{f \in S'(\mathbb{R}^n) : \; w_\gamma f \in A^s_{p,q}\}.$$

Preceding theory can be shifted from the unweighted spaces $A^s_{p,q}$ to the weighted spaces $A^s_{p,q}(\gamma)$. Monotonicity of $A^s_{p,q}(\gamma)$ both in s and γ, combined with related embeddings proves indicated decay.

Theorem 8. Let $2 \le n \in \mathbb{N}$. Let $1 \le p, q \le \infty$ ($p < \infty$ for F-spaces). Let $s > \max\left(1, \frac{n}{p}\right)$,

$$\gamma \ge 0 \quad \text{if} \quad 1 \le p \le 2 \qquad \text{and} \qquad \gamma > n\Big(\frac{1}{2} - \frac{1}{p}\Big) \quad \text{if} \quad 2 < p \le \infty.$$

Let

$$u_0 \in A^s_{p,q}(\gamma), \qquad v_0 \in A^{s+2}_{p,q}(\gamma)$$

be real and non-negative, $u_0(x) \ge 0$, $v_0(x) \ge 0$, $x \in \mathbb{R}^n$. Then the solutions $u(x,t)$, $v(x,t)$, $0 \le t < T$ are also real and non-negative,

$$u(x,t) \ge 0, \quad v(x,t) \ge 0, \qquad x \in \mathbb{R}^n, \quad 0 < t < T.$$

Remark 9. Basically L_2-proof, based on embeddings of $A^s_{p,q}(\gamma)$.

Theorem 10. Let $2 \le n \in \mathbb{N}$. Let $1 \le p, q \le \infty$ ($p < \infty$ for F-spaces). Let $s > \max\left(1, \frac{n}{p}\right)$, $\gamma > n$ and

$$u_0 \in A^s_{p,q}(\gamma), \qquad v_0 \in A^{s+2}_{p,q}(\gamma).$$

Let $u(x,t)$, $v(x,t)$, $0 \le t < T$ be the solutions where again $\alpha \ge 0$. Then

$$\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x < \infty, \qquad 0 < t < T,$$

and

$$\int_{\mathbb{R}^n} v(x,t)\,\mathrm{d}x = e^{-\alpha t}\int_{\mathbb{R}^n} v_0(x)\,\mathrm{d}x + c_\alpha(t)\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x < \infty, \qquad 0 < t < T,$$

with

$$c_\alpha(t) = \begin{cases} t, & \text{if } \alpha = 0, \\ \alpha^{-1}(1 - e^{-\alpha t}), & \text{if } \alpha > 0. \end{cases}$$

Remark 11. Conservation of the total cell mass. Influence of the damping constant $\alpha > 0$. Recall: $v_0 = 0$ auto-attractant.