Cocompactness and profile decompositions in Besov and Triebel-Lizorkin spaces

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- Classical concentration compactness (P.L.Lions et al): Handles convergence in problems without compactness. $u_n \rightharpoonup u$ in $H^{1,p}(\Omega)$, $|\nabla u_n|^p dx \rightharpoonup |\nabla u|^p dx +$ singular measure. Get the singular measure equal zero.

- Profile decomposition theory (Sobolev spaces: Struwe 1984; Solimini 1995; weaker version by Gerard 1998 and Jaffard 1999; Hilbert spaces: Schindler & CT 2003; Besov and TL spaces: H. Koch 2010; Moser-Trudinger embedding: Adimurthi & CT 2014; Banach spaces: Solimini & CT, 2016). Given $u_n \rightharpoonup u$ in $H^{1,p}(\Omega)$, describe $u_n - u$ as a sum of decoupled blowups of the form $t_n^{\frac{N-p}{p}} w(t_n(x - x_0))$, $t_n \to \infty$, with a vanishing remainder. Find reasons for $w = 0$.

- Functional-analytic theory of concentration: covers concentration mechanisms other than Euclidean blowups.

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Profile decomposition

Theorem

[Schindler-Tintarev and Solimini versions] Let H be a Hilbert space [resp. $\dot{H}^{1,p}(\mathbb{R}^N)$, $N > p$,] equipped with a dislocation group D, $D = \{u \mapsto 2^{\frac{N-p}{p}j} u(2^j(\cdot - y)) : j \in \mathbb{Z}, y \in \mathbb{R}^N\}$. Let $u_k \in H$ [$u_k \in \dot{H}^{1,p}(\mathbb{R}^N)$] be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$, such that for a renumbered subsequence $g_k^{(1)} = id$,

$$g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0, \quad |j_k^{(m)} - j_k^{(n)}| + |y_k^{(m)} - y_k^{(n)}| \to \infty, \; n \neq m,$$

$$g_k^{(n)^{-1}} u_k \rightharpoonup w^{(n)}, \quad 2^{-\frac{N-p}{p} j_k^{(n)}} u_k(2^{j_k^{(n)}} \cdot + y_k^{(n)}) \rightharpoonup w^{(n)},$$

$$u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0, \quad u_k - \sum_{n \in \mathbb{N}} 2^{\frac{N-p}{p} j_k^{(n)}} w^{(n)}(2^{j_k^{(n)}}(\cdot - y_k^{(n)})) \overset{L^{p*}}{\to} 0.$$

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Moreover, the series $\sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}$ converges unconditionally and uniformly with respect to k, and

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \leq \liminf \|u_k\|^2, \quad \sum_{n\in\mathbb{N}} \int |\nabla w^{(n)}|^p dx \leq \liminf \int |\nabla u_k|^p dx.$$

- D-weak convergence: One says that $u_k \overset{D}{\rightharpoonup} u$ if for every sequence $g_k \in D$, $g_k(u_k - u) \rightharpoonup 0$.
- Co-compactness: D-weak convergence in $\dot{H}^{1,p}$ implies convergence in L^{p*}.
- Dislocation group: if $g_k \in D$, $g_k \not\rightharpoonup 0$, then g_k has a strongly (pointwise) convergent subsequence.

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Banach space version

- Delta-convergence (Lim, Kuczumow): $u_k \rightharpoondown u$: $d(u_k, u) \leq d(u_k, v) + o(1)$ for any v. In Hilbert space same as weak convergence.
- Asymptotically complete metric space: every bounded sequence has an asymptotic center. UC Banach spaces are asymptotically complete.

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- Delta-compactness theorem: every bounded sequence in an asymptotically complete metric space has a convergent subsequence.
- Opial condition (equivalent form for reflexive AC (in particular, UC) Banach spaces): $u_k \rightharpoondown u \Longleftrightarrow u_k \rightharpoonup u$.
- For UC Banach space with Opial condition, profile decomposition theorem repeats with modification only in the stability estimate: $\sum_{n\in\mathbb{N}} \delta(\frac{w^{(n)}}{\liminf \|u_n\|}) \leq 1$.

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Besov and Triebel-Lizorkin spaces

- Following an unpublished paper of Michael Cwikel:
- $\|u\|_{\dot{B}^{s,p,q}} = \left\| (\|2^{js} P_j u\|_{L^p})_{j\in\mathbb{Z}} \right\|_{\ell^q}$,
  $\|u\|_{\dot{F}^{s,p,q}} = \left\| \|(2^{js} P_j u)_{j\in\mathbb{Z}}\|_{\ell^q} \right\|_{L^p}$ where $P_j u = F^{-1}\varphi_0(2^{-j}\cdot)Fu$.

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- Delta-convergence in UC-US spaces allows equivalent characterization as $u_k \rightharpoondown u$ in $E \iff (u_k - u)^* \rightharpoonup 0$ in E^* where u^* is Frechet derivative of $\frac{1}{2}\|u\|^2$.
- Calculations of derivative of the norms above amount to coincidence of Delta- and weak convergence (via pointwise convergence).
- Embeddings $\dot{B}^{s,p,a} \hookrightarrow \dot{B}^{t,q,a}$ $s > t$, $\frac{N}{p} - s = \frac{N}{q} - t$, are cocompact with respect to the rescaling group with the normalizing exponent $N/p - s$.

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More cocompact embeddings

- $\dot{B}^{s,p,a} \hookrightarrow \dot{F}^{0,p_s^*,b}$, $0 < s < N/p$, $a < p_s^* := \frac{pN}{N-ps}$.

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Affine Sobolev inequality

- Gaoyong Zhang (1999,2002):
  $$J_p(u) := \left(\int_{S_1} \frac{dS_\omega}{\|\omega \cdot \nabla u\|_p^N}\right)^{-1/N} \geq C\|u\|_{p^*},\ 1 \leq p < \infty,$$
  invariance with respect to $SL(N)$.

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- $J_p(u) = \left(\frac{1}{(N-1)!}\int_{\mathbb{R}^N} e^{-\|\xi\cdot\nabla u\|_p} d\xi\right)^{-1/N}.$
- Let $A_{i,j}[u] = \int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx$. Then
  $J_2(u) = \omega_N^{-1/N}(\det A[u])^{1/2N} = \frac{\omega_N^{-1/N}}{\sqrt{N}} \min_{T\in SL(N)} \|\nabla(u\circ T)\|_2.$
- Q.-Z. Huang, A.-J. Li (2016): $C^{-1}\min_{T\in SL(N)}\|\nabla(u\circ T)\|_p \leq J_p(u) \leq C\min_{T\in SL(N)}\|\nabla(u\circ T)\|_p$
- Conclusion: sequences with the bounded J_p have the same profile decomposition as sequences in $\dot{H}^{1,p}$ up to compositions with unimodular matrices.

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- Conclusion: sequences with the bounded J_p have the same profile decomposition as sequences in $\dot{H}^{1,p}$ up to compositions with unimodular matrices.

Affine Sobolev inequality

- Gaoyong Zhang (1999,2002):
  $J_p(u) := \left(\int_{S_1} \frac{dS_\omega}{\|\omega \cdot \nabla u\|_p^N} \right)^{-1/N} \geq C\|u\|_{p^*}$, $1 \leq p < \infty$,
  invariance with respect to $SL(N)$.

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