Eigenvalues and entropy numbers of powers of operators

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A refined Riesz theory

Spectral radius formula

- The spectrum $\sigma(T)$ of T on a complex Banach space X

$$\sigma(T) := \left\{ \lambda \in \mathbb{C} \ : \ \lambda I_X - T \ \text{is not invertible in} \ L(X) \right\}$$

- $\sigma(T)$ is contained in the circle of radius

$$\lim_{m \to \infty} \|T^m\|^{1/m}$$

with 0 as centre

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- $\sigma(T)$ is not empty and $\rho(T) = \mathbb{C}\backslash\sigma(T)$ is open

- Gelfand's spectral radius formula

$$r(T) := \sup_{\lambda\in\sigma(T)} |\lambda| = \lim_{m\to\infty} \|T^m\|^{1/m}$$

Eigenvalues of a compact operator

- A value λ is called an eigenvalue of T if there is $x \neq 0$ in X with

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Eigenvalues of a compact operator

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- $\dim N(\lambda I - T)$ is called the geometric multiplicity of $\lambda \neq 0$
- $\dim N_\infty(\lambda I_X - T)$, where

$$N_\infty(\lambda I_X - T) := \bigcup_{n=1}^{\infty} N\big((\lambda I_X - T)^n\big)$$

  is called the algebraic multiplicity of $\lambda \neq 0$

The essential spectrum

Definition

$S \in L(X, Y)$ is called a Fredholm operator if

$$\dim N(S) < \infty \quad \text{and} \quad \operatorname{codim} R(S) < \infty$$

- The essential spectrum $\sigma_{ess}(T)$ of T

$$\sigma_{ess}(T) := \left\{ \lambda \in \mathbb{C} \;\; : \;\; \lambda I_X - T \;\text{ is not Fredholm } \right\}$$

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- $S \in L(X)$ is a Fredholm operator $\iff$ its equivalence class $\overline{S}$ is invertible in the Calkin algebra $L(X)/K(X)$.

$$\sigma_{ess}(T) = \sigma(\overline{T})$$

The essential spectral radius

- The essential spectral radius

$$r_{ess}(T) := \sup_{\lambda \in \sigma_{\boldsymbol{ess}}(T)} |\lambda| = \lim_{m \to \infty} \|T^m\|_{ess}^{1/m}$$

- The Riesz part of the spectrum

$$\Lambda(T) := \{\, \lambda \in \sigma(T) \ : \ |\lambda| > r_{ess}(T) \,\}$$

is at most countable and consists of isolated eigenvalues of finite algebraic multiplicity.

Eigenvalue sequence for an arbitrary operator (1)

- The Riesz part of the spectrum

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Eigenvalue sequence for an arbitrary operator (1)

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$$\Lambda(T) := \{ \lambda \in \sigma(T) \ : \ |\lambda| > r_{ess}(T) \}$$

We assign an eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ for $T \in L(X)$ from the elements of the set $\Lambda(T) \cup \{r_{ess}(T)\}$ as follows:

- The eigenvalues are arranged in an order of non-increasing absolute values.
- Every eigenvalue $\lambda \in \Lambda(T)$ is counted according to its algebraic multiplicity.
- If T possesses less than n eigenvalues λ with $|\lambda| > r_{ess}(T)$, we let

$$\lambda_n(T) = \lambda_{n+1}(T) = \ldots = r_{ess}(T)$$

Eigenvalue sequence for an arbitrary operator (2)

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Remark

$$r(T) = |\lambda_1(T)| \quad \text{and} \quad r_{ess}(T) = \lim_{n \to \infty} |\lambda_n(T)|$$

Entropy numbers

Entropy numbers

Definition

The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X, Y)$ is defined by

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;:\; T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

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- Entropy numbers are: monotonne

$$0 \leqslant \ldots \leqslant \varepsilon_3(T) \leqslant \varepsilon_2(T) \leqslant \varepsilon_1(T) = \|T\|$$

- sub-multiplicative

$$\varepsilon_{kl}(RS) \leqslant \varepsilon_k(R)\,\varepsilon_l(S) \quad \text{for} \quad S \in L(X,Z) \quad \text{and} \quad R \in L(Z,Y)$$

- sub-additive

$$\varepsilon_{kl}(T_1 + T_2) \leqslant \varepsilon_k(T_1) + \varepsilon_l(T_2) \quad \text{for} \quad T_1, T_2 \in L(X,Y).$$

Measure of non-compactness

- The measure of non-compactness

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T) \qquad \text{and} \qquad \beta(T) \leqslant \|T\|_{ess}$$

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Theorem (1970, Nussbaum)

Let X be a complex Banach space and $T \in L(X)$, then

$$r_{ess}(T) = \lim_{m\to\infty} \beta(T^m)^{1/m}$$

R. D. Nussbaum
The radius of the essential spectrum
Duke Math. J. 37 (1970), 473–478.

Carl-Triebel's inequality (1980)

Let $\{\lambda_n(T)\}_{n=1}^{\infty}$ be an eigenvalue sequence of $T \in K(X)$ on a complex Banach space X.

- Carl-Triebel's inequality

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

B. Carl and H. Triebel
Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces
Math. Ann. 251 (1980), 129–133.

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B. Carl and H. Triebel
Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces
Math. Ann. 251 (1980), 129–133.

- Zemánek proved the inequality for arbitrary $T \in L(X)$

J. Zemánek
The essential spectral radius and the Riesz part of the spectrum
Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai 35 (1983), 1275–1289.

Interpolation

Banach couple

- We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

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For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1} \{\|a_0\|_{A_0} + \|a_1\|_{A_1}\}$$

Interpolation functor

- By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that

$$T|_{A_j} \in L(A_j, B_j), \quad j = 0, 1$$

Interpolation functor

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Definition

By an interpolation functor we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$

- $A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1$ for any $\vec{A} \in \vec{\mathcal{B}}$
- $T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B})$ for any $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and $T\colon \vec{A} \to \vec{B}$

Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \max\left\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\right\}$$

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$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \max\left\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\right\}$$

If in addition there exists $\theta \in (0,1)$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \, \|T\|_{A_0\to B_0}^{1-\theta} \, \|T\|_{A_1\to B_1}^{\theta},$$

then $\mathcal{F}$ is called of exponential type of θ.

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are of exponential type of θ.

Recollect...

- The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X,Y)$

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \ : \ T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

- The measure of non-compactness

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$$

Interpolation of the measure of non-compactness β

A delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$\beta\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\,\beta\big(T\colon A_0 \to B_0\big)^{1-\theta}\beta\big(T\colon A_1 \to B_1\big)^{\theta} \text{ ?}$$

Interpolation of the measure of non-compactness β (cont.)

- This question was answered positively for the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$

F. Cobos, P. Fernández-Martínez and A. Martínez
Interpolation of the measure of non-compactness by the real method
Studia Math. 135 (1999), 25–38.

P. Fernández-Martínez
Interpolation of the measure of non-compactness between quasi-Banach spaces
Rev. Mat. Complut. 19 (2006), 477–498.

R. Szwedek
Measure of non-compactness of operators interpolated by the real method
Studia Math. 175 (2006), 157–174.

Interpolation of the measure of non-compactness β (cont.)

- This question was answered positively for the complex interpolation functor $\mathcal{F}(\cdot) = [\cdot]_\theta$ in the case where $\vec{B}$ satisfies an approximation condition

F. Teixeira and D. E. Edmunds

Interpolation theory and measures of noncompactness

Math. Nachr. 104 (1981), 129–135.

R. Szwedek

On interpolation of the measure of non-compactness by the complex method

Q. J. Math. 66 (2015), 323–332.

Interpolation of entropy numbers fails

A more delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T \colon \vec{A} \to \vec{B}$

$$\varepsilon_{k_0 k_1}\big(T \colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, \varepsilon_{k_0}(T \colon A_0 \to B_0)^{1-\theta}\, \varepsilon_{k_1}(T \colon A_1 \to B_1)^{\theta} \text{ ?}$$

- This question was answered negatively for the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$

D. E. Edmunds and Yu. Netrusov
Entropy numbers and interpolation
Math. Ann. 351 (2011), 963–977.

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The reduction

$$\vec{B} = \vec{A}$$

M. Mastyło and R. Szwedek
Eigenvalues and entropy moduli of operators in interpolation spaces
J. Geom. Anal. 27 (2017), 1131–1177.

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$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;:\; T(U_X) \subset \bigcup_{i=1}^{n} \{x_i + \varepsilon U_X\}, \quad x_i \in X \right\}$$

- Carl's inequality

$$|\lambda_n(T)| \leqslant \sqrt{2}\, e_n(T) \quad \text{where} \quad e_n(T) := \varepsilon_{2^{n-1}}(T)$$

- Carl-Triebel's inequality

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

Interpolation variant of Carl-Triebel's inequality

Theorem (2017, Mastyło, Szwedek)

Suppose that $\mathcal{F}$ is an interpolation functor of exponential type of θ. If $T\colon \vec{A} \to \vec{A}$, then

$$\left|\lambda_n\Big(T|_{\mathcal{F}(\vec{A})}\Big)\right| \leqslant 2e_n(T|_{A_0})^{1-\theta}\, e_n(T|_{A_1})^{\theta}$$

and

$$\left(\prod_{i=1}^{n}\left|\lambda_i\Big(T|_{\mathcal{F}(\vec{A})}\Big)\right|\right)^{1/n} \leqslant \inf_{k_0,k_1\in\mathbb{N}} (k_0k_1)^{1/2n}\, \varepsilon_{k_0}(T|_{A_0})^{1-\theta}\, \varepsilon_{k_1}(T|_{A_1})^{\theta}$$

Stay motivated !

Generalizations of the spectral radius formula (1)

- Gelfand's spectral radius formula

$$|\lambda_1(T)| = \lim_{m\to\infty} \|T^m\|^{1/m}$$

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Definition

Given $T \in L(X)$, the n-th *approximation number* is defined by

$$a_n(T) := \inf\Big\{ \|T - S\| \ : \ S \in L(E, F) \ \text{with} \ \operatorname{rank}(S) < n \Big\}$$

- König's (1978) formula; a generalization for higher eigenvalues

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n(T^m)^{1/m}$$

H. König
A formula for the eigenvalues of a compact operators
Studia Math. 65 (1979), 141–146.

Generalizations of the spectral radius formula (2)

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

Generalizations of the spectral radius formula (2)

- The n-th entropy modulus $g_n(T)$ of $T \in L(X)$ is given by

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T) =: g_n(T)$$

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- Makai-Zemánek's formula (1982)

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} = \lim_{m \to \infty} g_n(T^m)^{1/m}$$

E. Jr. Makai and J. Zemánek
Geometrical means of eigenvalues
J. Operator Theory 7 (1982), 173–178.

Spectral entropy numbers (1)

Problem

In what form does it exist a formula for the spectral radius of T using the entropy numbers of powers of operators?

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Theorem (2017, Mastyło, Szwedek)

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\sup_{n\in\mathbb{N}} k^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m}$$

Definition

We define the k-th *spectral entropy number* $\mathcal{E}_k(T)$ by

$$\mathcal{E}_k(T) := \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m}$$

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$$\mathcal{E}_k(T) := \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m} \leqslant \varepsilon_k(T)$$

Spectral entropy numbers (2)

Theorem (2017, Mastyło, Szwedek)

Fix $t \in [1, \infty)$. *If* $\{t_m\} \subset \mathbb{N}$ *is such that* $\lim_{m\to\infty} t_m^{1/m} = t$, *then*

$$\sup_{n\in\mathbb{N}} t^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

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Definition

Define the *spectral entropy map* $t \mapsto \mathcal{E}_t(T)$ of T as follows

$$\mathcal{E}_t(T) := \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

Spectral entropy numbers (3)

Proposition (2017, Mastyło, Szwedek)

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\lim_{m\to\infty} \varepsilon_n(T^m)^{1/m} = \mathcal{E}_1(T) = r(T) \text{ for each } n \in \mathbb{N}$$

$$\lim_{m\to\infty} \varepsilon_m(T^m)^{1/m} = \mathcal{E}_1(T)$$

$$\lim_{m\to\infty} e_m(T^m)^{1/m} = \mathcal{E}_2(T) = \sup_{n\in\mathbb{N}} \left(\frac{\prod_{i=1}^{n} |\lambda_i(T)|}{\sqrt{2}} \right)^{1/n}$$

$$\mathcal{E}_\infty(T) := \lim_{t\to\infty} \mathcal{E}_t(T) = r_{ess}(T)$$

Spectral entropy numbers (4)

Proposition (2017, Mastyło, Szwedek)

Let X, Y be a complex Banach space.

- *If $R, S \in L(X)$ are commuting operators, then*

$$\mathcal{E}_{tu}(RS) \leqslant \mathcal{E}_t(R)\,\mathcal{E}_u(S)\,, \quad t, u \in [1, \infty].$$

-

$$\mathcal{E}_{t^n}(R^n) = \mathcal{E}_t(R)^n\,, \quad t \in [1, \infty],\ n \in \mathbb{N}.$$

- *If $T \in L(X, Y)$ and $U \in L(Y, X)$, then*

$$\mathcal{E}_t(TU) = \mathcal{E}_t(UT)\,, \quad t \in [1, \infty].$$

Entropy moduli

Definition

Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_n \quad (T)$ as follows

$$g_n \quad (T) := \inf_{k \in \mathbb{N}} k^{1/(2n)} \quad \varepsilon_k(T) \quad , \qquad n \in \mathbb{N}$$

Entropy moduli

Definition

Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_s(T)$ as follows

$$g_s(T) := \inf_{k \in \mathbb{N}} k^{1/(2s)} \varepsilon_k(T) \quad , \quad s \in (0, \infty)$$

Definition

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a sub-multiplicative function.
Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_{s,\varphi}(T)$ as follows

$$g_{s,\varphi}(T) := \inf_{k\in\mathbb{N}} k^{1/(2s)}\varphi\left(\varepsilon_k(T)\right), \quad s \in (0,\infty)$$

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$$g_{s,\varphi}(T) := \inf_{k\in\mathbb{N}} k^{1/(2s)}\varphi\left(\varepsilon_k(T)\right), \quad s \in (0,\infty)$$

- Denote by $\widetilde{\varphi}$ the function on $[0,\infty)$ given by

$$\widetilde{\varphi}(u) := \lim_{m\to\infty} \varphi(u^m)^{1/m}, \quad u \geqslant 0$$

- $\widetilde{\varphi}$ is sub-multiplicative and $\widetilde{\varphi} \leqslant \varphi$

Makai-Zemánek's formula revisited

Theorem (2017, Mastyło, Szwedek)

Let X be an arbitrary complex Banach space and $T \in L(X)$. Assume that $\varphi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing, sub-multiplicative and right-continuous function. Then

$$\inf_{t\in[1,\infty)} t^{1/(2s)}\widetilde{\varphi}(\mathcal{E}_t(T)) = \lim_{m\to\infty} g_{s,\varphi}(T^m)^{1/m}, \quad s \in (0,\infty)$$

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In particular,

$$\inf_{t\in[1,\infty)} t^{1/(2n)}\mathcal{E}_t(T) = \left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n}$$

Interpolation of spectral entropy numbers holds

Theorem (2017, Mastyło, Szwedek)

If $\mathcal{F}$ be an interpolation functor of exponential type of θ, then for any $T\colon \vec{A} \to \vec{A}$

$$\mathcal{E}_{k_0 k_1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{A})\big) \leqslant \mathcal{E}_{k_0}(T\colon A_0 \to A_0)^{1-\theta}\, \mathcal{E}_{k_1}(T\colon A_1 \to A_1)^{\theta}$$

Thank you for your attention!

Any questions?

R. Szwedek

Measure of non-compactness of operators interpolated by the real method

Studia Math. 175 (2006), 157–174.

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On interpolation of the measure of non-compactness by the complex method

Q. J. Math. 66 (2015), 323–332.

M. Mastyło and R. Szwedek

Eigenvalues and entropy moduli of operators in interpolation spaces

J. Geom. Anal. 27 (2017), 1131–1177.