Embedding of classical Lorentz type spaces involving weighted integral means

Filip Soudský

Faculty of Economics, University of South Bohemia, České Budějovice

Będlewo, September 21, 2017

Overview

1. Motivation
2. Main results
3. Some techniques and examples

Joint work with Amiran Gogatishvili, Martin Křepela and Luboš Pick.
Based on the recent paper see [6]

Joint work with Amiran Gogatishvili, Martin Křepela and Luboš Pick.
Based on the recent paper see [6]
In our talk, **weights** mean positive (strictly) locally integrable functions, we shall also use the notion of **non-increasing rearrangement** (see picture on the blackboard)

Joint work with Amiran Gogatishvili, Martin Křepela and Luboš Pick.
Based on the recent paper see [6]

In our talk, **weights** mean positive (strictly) locally integrable functions, we shall also use the notion of **non-increasing rearrangement** (see picture on the blackboard)

Definition Let $p, m \in (0, \infty)$ and let u, w be a couple of weights, define a functional $\| \cdot \|_{\Gamma^{p,m}(u,v)} : \mathcal{M}(\mathcal{R}) \to [0, \infty]$

$$\|f\|_{\Gamma^{p,m}(u,w)} := \left(\int_0^\infty \left(\int_0^t f^*(s)^p u(s) \mathrm{d}s \right)^{\frac{m}{p}} w(t) \mathrm{d}t \right)^{\frac{1}{m}}$$

Grand Lebesgue Spaces

Definition Let $p \in (0, \infty)$ the norm in **Grand Lebesgue space** is defined by

$$\|u\|_{p)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|u\|_{p-\varepsilon}.$$

We see that for any $q < p$ on a bounded domain one has

$$L^p \hookrightarrow L^{p)} \hookrightarrow L^q$$

Grand Lebesgue Spaces

Definition Let $p \in (0, \infty)$ the norm in **Grand Lebesgue space** is defined by

$$\|u\|_{p)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|u\|_{p-\varepsilon}.$$

We see that for any $q < p$ on a bounded domain one has

$$L^p \hookrightarrow L^{p)} \hookrightarrow L^q$$

Exercise(tough one) Find all $q > p$ for which the following holds

$$L^{p,q} \hookrightarrow L^{p)}, \forall q > p.$$

Grand Lebesgue Spaces

Definition Let $p \in (0, \infty)$ the norm in **Grand Lebesgue space** is defined by

$$\|u\|_{p)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|u\|_{p-\varepsilon}.$$

We see that for any $q < p$ on a bounded domain one has

$$L^p \hookrightarrow L^{p)} \hookrightarrow L^q$$

Exercise(tough one) Find all $q > p$ for which the following holds

$$L^{p,q} \hookrightarrow L^{p)}, \forall q > p.$$

The associated space to Grand Lebesgue space is called **small Lebesgue space** (denote by $L^{(p}$). By duality properties of Lebesgue spaces one easily observes

$$L^q \hookrightarrow (L^{p)})' \hookrightarrow L^{p'}$$

for any $q < p'$.

For the original motivation of Grand Lebesgue spaces see [7].

For the original motivation of Grand Lebesgue spaces see [7].
The Grand Lebesgue spaces were further investigated in [2, 3]. In the second one the following theorem was proved.

Relationship of Grand Lebesgue spaces and Lorentz spaces

Theorem

Let $p \in (1, \infty)$ then the following equivalences holds

(i)

$$\|u\|_{(p} \approx \int_0^1 \left(\int_0^t u^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t \log^{\frac{1}{p}}(\frac{1}{t})}.$$

(ii)

$$\|u\|_{p)} \approx \sup_{0<t<1} (1 - \log(t))^{\frac{-1}{p}} \left(\int_t^1 f^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}}$$

Relationship of Grand Lebesgue spaces and Lorentz spaces

Theorem

Let $p \in (1, \infty)$ then the following equivalences holds

(i)

$$\|u\|_{(p} \approx \int_0^1 \left(\int_0^t u^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} \frac{\mathrm{d}t}{t \log^{\frac{1}{p}}(\frac{1}{t})}.$$

(ii)

$$\|u\|_{p)} \approx \sup_{0<t<1} (1 - \log(t))^{\frac{-1}{p}} \left(\int_t^1 f^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}}$$

Another motivation comes from interpolation between Grand and small Lebesgue spaces. By interpolation we obtain certain cases of quantity $\|u\|_{\Gamma^{p,m}(u,w)}$. Details were given by Amiran.

Regularity of PDE solutions

Let $\Omega \subset \mathbb{R}^n$ be a domain of finite measure (WLOG $|\Omega| = 1$), and let us consider PDE in the form of

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x), x \in \Omega$$

The spaces $\Gamma^{p,m}(u, v)$ play role in study of regularity of solutions. Details are given in [1]

Embeddings between GΓ spaces

Notation Let $p, m \in (0, \infty)$ and let u, w be weights (positive measurable functions) define a functional $\|\cdot\|_{\Gamma^{p,m}(u,w)} : \mathfrak{M}(\mathcal{R}) \to [0, \infty]$ by

$$\|f\|_{\Gamma^{p,m}(u,w)} := \left(\int_0^\infty \left(\int_0^t f^*(s)^p u(s) \mathrm{d}s \right)^{\frac{p}{m}} w(s) \mathrm{d}t \right)^{\frac{1}{m}}$$

Let $E \subset \mathcal{R}$ be a set of measure t denote

$$\varphi_i(t) = \|\chi_E\|_{\Gamma^{p_i,m_i}(u_i,w_i)}$$

(in the case of some reasonable setting it would be the fundamental function of a Banach function space)
Denote

$$\sigma(t) := \frac{U_1(t)^{\frac{p_1^2}{m_1(p_1-m_2)}-1} u_1(t) \int_0^t U_1^{\frac{p_1}{m_1}}(s) w(s) \mathrm{d}s \int_t^\infty w_1(s) \mathrm{d}s}{\varphi_1(t)}$$

Technical assumptions

- $$\int_0^\infty u_1 = \infty$$

Technical assumptions

- $$\int_0^\infty u_1 = \infty$$

- $$\varphi_1(t) < \infty, \quad \int_0^t \sigma < \infty$$

Technical assumptions

- $$\int_0^\infty u_1 = \infty$$

- $$\varphi_1(t) < \infty, \quad \int_0^t \sigma < \infty$$

- $$\int_t^\infty U_1^{\frac{m}{p}} w_1 = \int_t^\infty w_1 = \int_t^\infty \sigma = \infty$$

Technical assumptions

- $$\int_0^\infty u_1 = \infty$$

- $$\varphi_1(t) < \infty, \quad \int_0^t \sigma < \infty$$

- $$\int_t^\infty U_1^{\frac{m}{p}} w_1 = \int_t^\infty w_1 = \int_t^\infty \sigma = \infty$$

- $$\int_0^1 U_1^{\frac{p_1 m_2}{m_1(p_1 - m_2)}} \sigma = \infty$$

Theorem (simplest case only)

Let $p_2 < m_2$, $m_1 < m_2$, $p_1 < p_2$, $p_i, m_i \in (1, \infty)$. *Let us denote by* C *the optimal constant of inequality*

$$\|f\|_{\Gamma^{p_1,m_1}(u_1,w_1)} \leq C\|f\|_{\Gamma^{p_2,m_2}(u_2,w_2)} \tag{1}$$

then the equivalence

$$C \approx \sup_{t>0} \frac{\varphi_2(t)}{\varphi_1(t)},$$

holds.

Theorem (simplest case only)

Let $p_2 < m_2$, $m_1 < m_2$, $p_1 < p_2$, $p_i, m_i \in (1, \infty)$. *Let us denote by* C *the optimal constant of inequality*

$$\|f\|_{\Gamma^{p_1,m_1}(u_1,w_1)} \leq C\|f\|_{\Gamma^{p_2,m_2}(u_2,w_2)} \tag{1}$$

then the equivalence

$$C \approx \sup_{t>0} \frac{\varphi_2(t)}{\varphi_1(t)},$$

holds.

curiosity The whole theorem takes 7 pages.

Examples

- $$\Gamma^{1,m}_{1,v} = \Gamma^m(t^{\frac{1}{m}} w)$$

Examples

- $$\Gamma^{1,m}_{1,v} = \Gamma^m(t^{\frac{1}{m}} w)$$

- If $p = m$ easy calculation involving Fubini theorem shows

$$\left(\int_0^\infty \int_0^t f^*(s)^m u(s) \mathrm{d}s w(t) \mathrm{d}t\right)^{\frac{1}{m}} = \left(\int_0^\infty f^*(s)^m u(s) \int_s^\infty w(t) \mathrm{d}t \mathrm{d}s\right)^{\frac{1}{m}}$$

Therefore $\Gamma^{m,m}(u,w) = \Lambda^m(v)$, where $v = u \int_t^\infty w$

Techniques used in the proof

- Duality approach

Techniques used in the proof

- Duality approach
- Hardy type inequalities. Some of these were known from [5] and [4]

Posible modifications, and further questions

- The case of $p_2 > m_2$ and the cases with $p_i, m_i < 1$

Posible modifications, and further questions

- The case of $p_2 > m_2$ and the cases with $p_i, m_i < 1$
- Investigation of functional properties of $\Gamma^{p,m}(u, w)$

[Bibliography]

A. Alberico, A. Cianchi, and C. Sbordone.
Continuity properties of solutions to the *p*-Laplace system.
Adv. Calc. Var., 10(1):1–24, 2017.

A. Fiorenza.
Duality and reflexivity in grand Lebesgue spaces.
Collect. Math., 51(2):131–148, 2000.

A. Fiorenza and GE Karadzhov.
Grand and small lebesgue spaces and their analogs.
Zeitschrift für Analysis und ihre Anwendungen, 23(4):657–681, 2004.

A. Gogatishvili, B. Opic, and L. Pick.
Weighted inequalities for hardy-type operators involving suprema.
Collect. Math., 57(3):227–255, 2006.

A. Gogatishvili and L. Pick.
Discretization and anti-discretization of rearrangement-invariant norms.

Publ. Mat., 47(2):311–358, 2003.

A. Gogatishvili, M. Křepela, L. Pick, and F. Soudský.
Embeddings of lorentz-type spaces involving weighted integral means.
Journal of Functional Analysis, 273(9):2939 – 2980, 2017.

T. Iwaniec and C. Sbordone.
On the integrability of the Jacobian under minimal hypotheses.
Arch. Rational Mech. Anal., 119(2):129–143, 1992.