Function spaces via hyperbolic fillings

Tomás Soto (University of Jyväskylä)

Joint work with Mario Bonk and Eero Saksman

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$\dot{M}^{1,p}(Z)$, where $\frac{Q}{Q+1} < p < \infty$.

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Define the graph (X, E) so that $X := \bigsqcup_n X_n$ and $x \in X$ and $x' \in X$ are connected by an edge in E if and only if $|\ell(x) - \ell(x')| \leq 1$ and $B(x) \cap B(x') \neq \emptyset$.

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Denote the endpoints of $e \in E$ by $e_- \in X$ and $e_+ \in X$.
Under some mild additional assumptions on Z, it turns out that (X, E) endowed with the natural path metric is a hyperbolic space in the sense of Gromov, whose boundary at infinity coincides with Z. Hence the name.

Function spaces

Definition

$\dot{\mathcal{F}}^s_{p,q}(Z)$ is defined as the quasi-normed space of functions in $L^1_{\mathrm{loc}}(Z)$ (or rather, equivalence classes modulo additive a.e. -everywhere constants) such that

$$\|f\|_{\dot{\mathcal{F}}^s_{p,q}(Z)} := \left\| \left(\fint_{B(e_+)} f d\mu - \fint_{B(e_-)} f d\mu \right)_{e \in E} \right\|_{\mathcal{J}^s_{p,q}(E)}$$

is finite, where

$$\|u\|_{\mathcal{J}^s_{p,q}(E)} := \left(\int_Z \left(\sum_{e \in E} 2^{|e_-|sq} |u(e)|^q \chi_{B(e_+) \cup B(e_-)}(\xi) \right)^{p/q} d\mu(\xi) \right)^{1/p}.$$

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All spaces above are quasi-Banach spaces, and all quasinorms above are independent of the choice of the hyperbolic filling, up to multiplicative constants.

Function spaces

This definition makes sense:

Proposition

We have

$$\dot{\mathcal{F}}^s_{p,q}(Z) = \dot{M}^s_{p,q}(Z) \quad \textit{and} \quad \dot{\mathcal{B}}^s_{p,q}(Z) = \dot{N}^s_{p,q}(Z)$$

for all admissible values of parameters in the smoothness range $0 < s < 1$.

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Furthermore, the function spaces are obtained as *retracs* of the corresponding sequence spaces:

Proposition

For all admissible parameters in the smoothness range $0 < s < 1$*, there exist bounded linear operators*

$$\mathcal{S}\colon \dot{\mathcal{F}}^s_{p,q}(Z) \to \mathcal{J}^s_{p,q}(E) \quad \textit{and} \quad \mathcal{R}\colon \mathcal{J}^s_{p,q}(E) \to \dot{\mathcal{F}}^s_{p,q}(Z)$$

such that $\mathcal{R} \circ \mathcal{S}$ *is the identity mapping on* $\dot{\mathcal{F}}^s_{p,q}(Z)$*.*

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such that $\mathcal{R} \circ \mathcal{S}$ *is the identity mapping on* $\dot{\mathcal{F}}^s_{p,q}(Z)$*. This also holds with* $\dot{\mathcal{B}}$ *and* $\mathcal{I}$ *in place of* $\dot{\mathcal{F}}$ *and* $\mathcal{J}$ *respectively.*

Quasiconformal invariance

A homeomorphism $\varphi\colon Z \to Z'$ between two metric spaces is said to be *quasiconformal* if

$$\sup_{\xi \in Z} \left(\limsup_{r \to 0} \frac{\sup\{|f(\eta) - f(\xi)| \,:\, |\eta - \xi| \leq r\}}{\inf\{|f(\eta) - f(\xi)| \,:\, |\eta - \xi| \geq r\}} \right) < \infty.$$

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Basic question: Which function spaces A are quasiconformally invariant in that

$$f \circ \varphi \in A \quad \text{and} \quad \|f \circ \varphi\|_A \leq C(\varphi)\|f\|_A$$

(in some sense) for all $f \in A$ whenever φ is quasiconformal?

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For all of these spaces, we have $s - d/p = 0$. This is to be expected, since

$$\|f(\lambda\cdot)\|_{\dot{A}^s_{p,q}(\mathbb{R}^d)} \approx \lambda^{s-d/p}\|f\|_{\dot{A}^s_{p,q}(\mathbb{R}^d)}$$

for all $\lambda > 0$.

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Theorem (Koskela-Yang-Zhou '11, Bonk-Saksman-S)

Let (Z, d, μ) *and* (Z', d', μ') *be Q-Ahlfors regular metric measure spaces* $(Q > 1)$ *with "controlled geometry" in the sense of Heinonen and Koskela, and suppose that* $0 < s < 1$ *and* $Q/(Q+s) < q \leq \infty$*. Then the composition operator induced by a homeomoprhism* $\varphi \colon Z \to Z'$ *is bounded from* $\dot{\mathcal{F}}^s_{Q/s,q}(Z')$ *to* $\dot{\mathcal{F}}^s_{Q/s,q}(Z)$ *if and only if* φ *is quasiconformal.*

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"Proof".

Sufficiency: a quasiconformal mapping $\varphi \colon Z \to Z'$ is known to extend as a *quasi-isometric* mapping Φ between the corresponding hyperbolic fillings. This means that Φ satisfies

$$\lambda^{-1} d_X(x, y) - c \leq d_{X'}\big(\Phi(x), \Phi(y)\big) \leq \lambda d_X(x, y) + c \quad \text{for all } x,\ y \in X$$

for some constants λ, $c > 0$.

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Necessity: Capacity estimates. □

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$$\mathcal{R}f = f_{|F} \quad \text{for all continuous } f \in A(Z)$$

and

$$\mathcal{R} \circ \mathcal{E} = \mathrm{id}_{B(F)}.$$

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where $0 < s < 1$, $\max(\frac{\gamma}{s}, \frac{Q}{Q-\gamma+s}) < p < \infty$ *and* $0 < q \leq \infty$.

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We also have

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All this also holds with $\mathcal{B}$, $\mathcal{F}$ *and* M *in place of* $\dot{\mathcal{B}}$, $\dot{\mathcal{F}}$ *and* $\dot{M}$.

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"Proof".

Choose the hyperbolic fillings so that $X^F \subset X^Z$ and $E^F \subset E^Z$. □

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- Complex interpolation.
- Density of Lipschitz functions with bounded support.
- Franke-Jawerth embeddings and results on pointwise multipliers (Yuan '17).

This talk is based on the following two papers.

M. Bonk, E. Saksman and T. Soto: *Triebel-Lizorkin spaces on metric spaces via hyperbolic fillings*, Indiana Univ. Math. J., to appear.

E. Saksman and T. Soto: *Traces of Besov, Triebel-Lizorkin and Sobolev spaces on metric spaces*, arXiv:1606.08729, submitted.

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- *W. Yuan: Besov and Triebel–Lizorkin spaces on metric spaces: Embeddings and pointwise multipliers, J. Funct. Anal. 453 (2017), no. 1, 434–457.*
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Thank you for your attention!