Approximation Numbers of Embeddings of Sobolev Spaces of Dominating Mixed Smoothness – Preasymptotics and Asymptotics

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Approximation numbers

- **Classical concept** (functional analysis)
- $T : X \to Y$, linear operator which maps the Banach space X into the Banach space Y
- **Approximation numbers**

$$a_n(T : X \to Y) := \inf_{\mathrm{rank}\, A < n} \sup_{\|x|X\| \le 1} \| Tx - Ax|Y\|$$
$$= \inf_{\mathrm{rank}\, A < n} \| T - A : X \to Y\|, \qquad n \in \mathbb{N}\,.$$

- Related to Kolmogorov numbers, Gelfand numbers, n-widths, s-numbers, entropy numbers, ...

A space of smooth functions

- Consider the class F_d of functions $f : [0,1]^d \to \mathbb{R}$ s.t.

$$\sup_\alpha \| D^\alpha f \|_\infty < \infty$$

- ... and L_∞-approximation

Theorem (Novak/Woźniakowski (2009))

For L_∞-approximation over F_d we have

$$a_n(Id : F_d \to L_\infty) = 1 \qquad \textit{for all } n = 0, 1, ..., 2^{\lfloor d/2 \rfloor} - 1 \,.$$

- Hinrichs, Novak, M. Ullrich and Woźniakowski (2014).
- Novak, Woźniakowski: *Tractability of Multivariate Problems*, I, II, III, EMS Tracts in Math. **6**, **12**, **18**.

Here:

- $T = Id$ - identity operator;
- X - periodic Sobolev space on $\mathbb{T}^d$;
- Y - $L_2(\mathbb{T}^d)$.

Sobolev spaces:

- $H^s(\mathbb{T}^d)$ - isotropic Sobolev space;
- $H^s_{\text{mix}}(\mathbb{T}^d)$ - Sobolev space of dominating mixed smoothness;
- $H^{\vec{s},\vec{q}}_{\text{mix}}(\mathbb{T}^d)$ - Sobolev space of anisotropic dominating mixed smoothness.

In some sense, the behaviour of $a_n(Id: \ X \to Y)$ is known in all these situations since fifty years, in particular

$$a_1(Id: \ X \to Y) = 1 > a_n(Id: \ X \to Y)\,, \qquad n > 1\,.$$

New: dependence on d !

Isotropic Sobolev spaces on $\mathbb{T}^d$

- **Sobolev spaces** on the torus $\mathbb{T}^d$, smoothness $m \in \mathbb{N}$,

$$\| f \,|H^m(\mathbb{T}^d)\| := \Big(\sum_{|\alpha|_1 \le m} \| D^\alpha f \,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2}$$

- **Equivalent norms**

$$\| f \,|H^m(\mathbb{T}^d)\|^* \quad := \quad \Big(\| f \,|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^d \Big\| \frac{\partial^m f}{\partial x_j^m} \Big| L_2(\mathbb{T}^d) \Big\|^2 \Big)^{1/2}$$

$$\| f \,|H^m(\mathbb{T}^d)\|^{**} \quad := \quad \max_{|\alpha|_1 \le m} \| D^\alpha f \,|L_2(\mathbb{T}^d)\|$$

Fourier coefficients

- $f \in L_2(\mathbb{T}^d)$, Fourier coefficients

$$c_k(f) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,, \qquad k \in \mathbb{Z}^d\,.$$

- Fourier series

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} c_k(f)\, e^{ikx}$$

- Fourier series of the derivative $D^\alpha f \in L_2(\mathbb{T}^d)$

$$D^\alpha f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} c_k(f)(ik)^\alpha\, e^{ikx}$$

Fractional Sobolev spaces $s > 0$

- Natural norm

$$\| f \,|H^s(\mathbb{T}^d)\|^\square = \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(1 + \sum_{j=1}^d |k_j|^2\Big)^s \Big]^{1/2}$$

- Modified natural norm

$$\| f \,|H^s(\mathbb{T}^d)\|^* = \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(1 + \sum_{j=1}^d |k_j|^{2\,s}\Big) \Big]^{1/2}$$

- This implies

$$\frac{1}{m!} \| f \,|H^{m,\square}(\mathbb{T}^d)\|^2 \le \| f \,|H^m(\mathbb{T}^d)\|^2 \le \| f \,|H^{m,\square}(\mathbb{T}^d)\|^2$$

Classical results for Sobolev spaces

- **A.N. Kolmogorov**, Über die beste Annäherung von Funktionen einer Funktionklasse, Ann. Math., 37(**1936**), 107–111,

$$a_n(Id : H^m(\mathbb{T}) \to L_2(\mathbb{T})) = n^{-m} .$$

- **J.W. Jerome** (1967) (non-periodic),
- **H.Triebel**, Interpolation theory, . . . , North-Holland, 1978 (non-periodic),
- **V.N. Temlyakov**, Approximation of periodic functions, Nova Science, 1993.

  Let $s > 0$. There exist positive constants $c_s(d)$, $C_s(d)$ such that

$$c_s(d)n^{-s/d} \leq a_n(Id : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq C_s(d)n^{-s/d}$$

  holds for all $n \in \mathbb{N}$.

Sobolev spaces of dominating mixed smoothness

$m \in \mathbb{N}$:

$$\| f \,|H^m_{\text{mix}}(\mathbb{T}^d)\| := \Big(\sum_{\substack{\alpha_j \le m \\ j=1,\ldots,d}} \| D^\alpha f \,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2}$$

$$H^m_{\text{mix}}(\mathbb{T}^d) \quad = \quad \underbrace{H^m(\mathbb{T}) \otimes \ldots \otimes H^m(\mathbb{T})}$$

m-fold tensor product of the univariate space $H^m(\mathbb{T})$

Embeddings:

$$H^{md}(\mathbb{T}^d) \hookrightarrow H^m_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^m(\mathbb{T}^d)\,.$$

Sobolev spaces of d.m. fractional order of smoothness

$$H^s_{\text{mix}}(\mathbb{T}^d) \quad = \quad \underbrace{H^s(\mathbb{T}) \otimes \ldots \otimes H^s(\mathbb{T})}$$

m-fold tensor product of the univariate space $H^s(\mathbb{T})$

$$\| \, f \, | H^s_{\text{mix}}(\mathbb{T}^d)\|^{\square} = \Big(\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^2)^s \Big)^{1/2}$$

$$\| \, f \, | H^s_{\text{mix}}(\mathbb{T}^d)\|^{*} = \Big(\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^{2s}) \Big)^{1/2}$$

$0 < q < \infty$:

$$\| \, f \, | H^{s,q}_{\text{mix}}(\mathbb{T}^d)\| = \Big(\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^q)^{2s/q} \Big)^{1/2}$$

- **K.I. Babenko** (1960);
- **B.S. Mitjagin** (1962).

Let $s > 0$. There exist positive constants $c_s(d)$, $C_s(d)$ such that

$$\begin{aligned} c_s(d)\, n^{-s} (\log n)^{(d-1)s} & \le a_n(Id : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ & \le C_s(d)\, n^{-s} (\log n)^{(d-1)s} \end{aligned}$$

holds for all $n \ge 2$.

Asymptotic behaviour

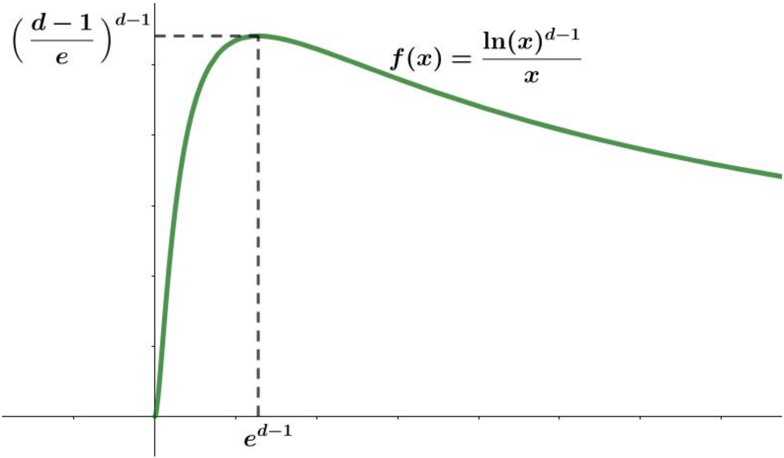
Theorem (KSU 2015)

Let $s > 0$, $0 < q < \infty$ *and* $d \in \mathbb{N}$. *We have*

$$\lim_{n\to\infty} \left[\frac{n}{(\log n)^{d-1}}\right]^s a_n(Id : H^{s,q}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \left(\frac{2^d}{(d-1)\,!}\right)^s \asymp \left(\frac{d}{2\pi}\right)^{s/2} \left(\frac{2e}{d}\right)^{ds}$$

- For $d > 5$ we have superexponential decay in d.
- Bungartz, Griebel (2004) (nonperiodic, spaces with boundary conditions, different norm).
- Schwab, Süli, Todor (2008, nonperiodic, different norm).
- Dinh Dung, Ullrich (2013), Chernov, Dinh Dung (2016) (different norm).

Asymptotic behaviour



Asymptotic behaviour

$$a_n(Id: H^{s,q}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \quad \leq \quad C_s(d) \underbrace{n^{-s}(\log n)^{(d-1)s}}_{\text{this is growing on } [2, e^{d-1}]}$$

But the numbers $a_n(Id: H^{s,q}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ are decreasing and ≤ 1 for all n.

$\Longrightarrow$ no information on the behaviour of the a_n on $[2, e^{d-1}]$!

For d large we can not deal with e^{d-1} information !

The preasymptotic behaviour

We need to know estimates of a_n with $n \leq e^{d-1}$! This we will call the **preasymptotic behaviour**.

Theorem (KSU 2015)

Let $s > 0$, $d \in \mathbb{N}$ and $1 < n \leq \frac{d}{2} 4^d$. Then one has

$$a_n(Id : H^{s,1}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{e^2}{n}\right)^{\frac{s}{2+\log_2 d}} .$$

- Bungartz, Griebel (2004, nonperiodic, boundary conditions, different norm);
- Schwab, Süli, Todor (2008, nonperiodic, different norm).

The preasymptotic behaviour

Two-sided estimates

$$\alpha(n,d) := 2 + \log_2\left(\frac{d}{\log_2 n} + \frac{1}{2}\right);$$
$$\beta(n,d) := 3 + \log_2\left(\frac{d}{\log_2 n}\, e\,(2 + \log_2(2ed))\right).$$

Theorem (KSU 2015/17)

Let $d \in \mathbb{N}$, $d \geq 2$, and $s > 0$. For all $2 \leq n \leq \frac{d}{2} 4^d$ it holds

$$2^{-s}(2n)^{-\frac{s}{\alpha(n,d)}} \leq a_n(Id : H^{s,1}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq (n)^{-\frac{s}{\beta(n,d)}}.$$

- There is still a gap !

Some comments

- We always know how to construct an optimal linear operator. This is an appropriate partial sum of the Fourier series (for given rank n we choose those $c_k(f)$ where $w(k)$ belongs to the group of the n largest values of w).
- Isotropic Sobolev spaces: KSU (2013), Kühn, Mayer, Ullrich (2017).
- Anisotropic Sobolev spaces: Jia Chen and Heping Wang (2017)
- Sobolev spaces on the sphere: Jia Chen and Heping Wang (2017)
- Periodic and nonperiodic Sobolev spaces of d.m.s. (more general frame): David Krieg (2017).
- Gevrey-Sobolev spaces: Kühn, Petersen (2017), Kühn, Mayer, Ullrich (2017).

The method

Let $F_d(w)$ be a Hilbert space of integrable functions on the d-dimensional torus $\mathbb{T}^d$ such that

$$f \in F_d(w) \iff \|f|F_d(w)\| := \Big(\sum_{k\in\mathbb{Z}^d} w(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty \, .$$

Here $w(k) > 0$, $k \in \mathbb{Z}^d$, are certain weights.
Let $(\sigma_n)_{n=1}^\infty$ be the non-increasing rearrangement of the sequence $(1/w(k))_{k\in\mathbb{Z}^d}$. Then

$$a_n(Id: \; F_d(w) \to L_2(\mathbb{T}^d)) = \sigma_n \, .$$

(See books of König, Pietsch, Pinkus, Novak/Woźniakowski, ...)

Combinatorial problem: rearrange tensor products of sequences !

Sobolev spaces of dominating mixed anisotropic smoothness

We need smaller spaces for large d!

Papageorgiou and Woźniakowski (2009): Different smoothness assumptions with respect to different directions.

$$H^{s_1}(\mathbb{T}) \otimes H^{s_2}(\mathbb{T}) \otimes \ldots \otimes H^{s_d}(\mathbb{T})$$

$$0 < s_1 \le s_2 \le \ldots \le s_d \, .$$

- Tractability questions.

Sobolev spaces of dominating mixed anisotropic smoothness

Definition

Let $\vec{s} = (s_1, \ldots, s_d)$, $\min_j s_j > 0$, *and let* $\vec{q} = (q_1, \ldots, q_d)$, $0 < q_j < \infty$. *The periodic anisotropic mixed Sobolev space* $H^{\vec{s},\vec{q}}_{\mathrm{mix}}(\mathbb{T}^d)$ *is the collection of all* $f \in L_2(\mathbb{T}^d)$ *such that*

$$\| f \, | H^{\vec{s},\vec{q}}_{\mathrm{mix}}(\mathbb{T}^d)\| := \Big[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^{d} \big(1 + |k_j|^{q_j}\big)^{2s_j/q_j} \Big]^{1/2} < \infty \, .$$

- $H^{\vec{s},\vec{q}}_{\mathrm{mix}}(\mathbb{T}^d) = H^{\vec{s},\vec{2}}_{\mathrm{mix}}(\mathbb{T}^d)$ in the sense of equivalent norms.
- $H^{\vec{s}}_{\mathrm{mix}}(\mathbb{T}^d)$: Mitjagin, Telyakovskij, Nikol'skaya, Tikhomirov, Galeev, Temlyakov, ...

$$0 < s_1 = \ldots = s_\nu < s_{\nu+1} \le \ldots \le s_d \, .$$

$$H^{s_1,q_1}_{\mathrm{mix}}(\mathbb{T}^\nu) \otimes H^{s_{\nu+1},q_{\nu+1}}(\mathbb{T}) \otimes \ldots \otimes H^{s_d,q_d}(\mathbb{T})$$

Lemma

Let $a := (a_k)_{k=1}^\infty \in c_0$ and $b := (b_m)_{m=1}^\infty \in \ell_1$ be two sequences of positive real numbers. By $c := (c_n)_{n=1}^\infty$ we denote the non-increasing rearrangement for the tensor product sequence $a \otimes b := (a_k \cdot b_m)$. Suppose $\alpha \ge 0$ and $\lim_{k\to\infty} \frac{k \cdot a_k}{(\log k)^\alpha} = 1$. Then

$$\lim_{n\to\infty} \frac{n \cdot c_n}{(\log n)^\alpha} = \| \, b \, \|_{\ell_1} \, .$$

Asymptotic behaviour

Theorem

Let $d \geq 2$ and $1 \leq \nu < d$. Then

$$\lim_{n\to\infty} \left[\frac{n}{(\log n)^{\nu-1}}\right]^{s_1} a_n(Id : H^{\vec{s},\vec{q}}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \left[\frac{2^\nu}{(\nu-1)!} \prod_{j=\nu+1}^{d} B_j\right]^{s_1},$$

where

$$B_j := 1 + 2\sum_{m=1}^{\infty} (1 + m^{q_j})^{-\frac{s_j}{s_1\, q_j}}, \qquad j = \nu+1, \ldots d.$$

- $s_j/s_1 > 1$, $j > \nu \quad \Longrightarrow \quad B_j < \infty$.
- Telyakovskii (1964):

$$a_n(Id : H^{\vec{s},\vec{2}}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \asymp n^{-s}(\log n)^{(\nu-1)s}.$$

The preasymptotic behaviour

Theorem

Let

$$0 < s_1 = s_2 = \ldots = s_\nu < ts_1 = s_{\nu+1} \leq \ldots \leq s_d$$

for some $t \geq 3$. *Further we suppose*

$$\frac{\nu}{d-\nu} \leq t\,.$$

Then, for all n, *satisfying*

$$\max\left(e^2\,(4d)^\nu, 3^\nu\left(\frac{t+1}{t-1}\right)^{d-\nu}\right) \leq n \leq 2^{d-\nu}(2^{d-\nu}-1)\,,$$

we have

$$a_n(Id:\ H^{\vec{s},\vec{1}}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq 3^{s_1\nu/2}\left(\frac{t+1}{t-1}\right)^{\frac{s_1(d-\nu)}{2}} n^{-s_1/2}\,.$$

Corollary

Let $1 \leq \nu < d$ *such that* $d - \nu \geq 3$. *Let*

$$0 < s_1 = s_2 = \ldots = s_\nu < ts_1 = s_{\nu+1} \leq \ldots \leq s_d$$

for some $t \geq d - \nu - 1$. *Further we suppose*

$$\frac{\nu}{d-\nu} \leq t\,.$$

Then, for all n, *satisfying*

$$e^2 \max((4d)^\nu, 3^{\nu+2}) \leq n \leq 2^{d-\nu}(2^{d-\nu} - 1)\,,$$

we have

$$a_n(Id : H^{\vec{s},\vec{1}}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq (3^{1+\nu/2}\, e)^{s_1}\, n^{-s_1/2}\,.$$

Theorem

Let

$$0 < s_1 = s_2 = \ldots = s_\nu < t s_1 = s_{\nu+1} \leq \ldots \leq s_d$$

for some $t \geq 3$. *Further we suppose*

$$\frac{\nu}{d-\nu} \leq t\,.$$

Then, for all n, *satisfying*

$$\max\left(e^2\,(4d)^\nu, 3^\nu\left(\frac{t+1}{t-1}\right)^{d-\nu}\right) \leq n \leq 2^{d-\nu}(2^{d-\nu}-1)\,,$$

we have

$$a_n(Id:\ H^{\vec{2s},\vec{2}}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq 3^{s_1\nu/2}\left(\frac{t+1}{t-1}\right)^{\frac{s_1(d-\nu)}{2}} n^{-s_1/2}\,.$$