Singular Integrals and a Problem on Mixing Flows

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I. Bressan’s open problem

II. A toy version

III. The Bianchini function space approach to Bressan’s problem –Reduction to a trilinear singular integral form.

IV. Christ-Journé multilinear (or just trilinear) singular integral forms

V. A Hardy space bound for a singular integral operator.

VI. Open problems.

Mixing at scale ε

Fix a constant $\kappa \ll 1/2$, say $\kappa = 1/3$ or $1/10$.

• **Definition:** A set $E \subset \mathbb{T}^d$ is mixed at (small) scale ε if for every ball B of radius ε

$$|E \cap B| \geq \kappa |B| \quad \text{and} \quad |E^{\complement} \cap B| \geq \kappa |B|$$

• Vague question:
Given a non-mixed set $E_0 \subset \mathbb{T}^d$, an incompressible flow $\Phi = (\Phi_t)$ with $\Phi_0 = Id$ what is the 'cost ' or 'work' for transforming E_0 to a set $\Phi_T E_0$ mixed at scale ε?

Bressan's open problem (2003)

Let $v : \mathbb{T}^d \times [0, T] \to \mathbb{T}^d$ be a time-dependent (a priori smooth) divergence free vector field on $\mathbb{T}^d$. Let Φ be the flow generated by v,

$$\frac{d}{dt}\Phi_t(x) = v(\Phi_t(x), t), \quad \Phi_0(x) = x\,.$$

Let $A_0 = \{x \in \mathbb{T}^d : 0 < x_1 < 1/2\}$. *Suppose* that at time T the set $\Phi_T(A_0)$ is mixed at scale ε. Is there $c > 0$, independent of v and ε, so that

$$\int_0^T \|Dv(\cdot, t)\|_1 \, dt \geq c \log(1/\varepsilon)\,?$$

• Motivated by work on flows of vector fields with weaker than Lipschitz regularity (DiPerna–P.Lions, Ambrosio, ...).

• Crippa-De Lellis result (2008):

$$\int_0^T \|M_{HL}[Dv(\cdot,t)]\|_1\, dt \geq c\log(1/\varepsilon)$$

i.e. $L^1(\mathbb{T}^d)$ is replaced by L^p, $p > 1$, or even $L\log L(\mathbb{T}^d)$.

• We'll discuss examples v_ε of mixing vector fields at scale ε with

$$\int_0^T \|Dv_\varepsilon(\cdot,t)\|_1\, dt \approx c\log(1/\varepsilon)$$

• Recent examples v_ε of mixing vector fields at scale ε, due to Yao and Zlatoš and to Alberti, Crippa, and Mazzucato have for $1 < p < \infty$

$$\int_0^T \|Dv_\varepsilon(\cdot,t)\|_p\, dt \approx \log(1/\varepsilon).$$

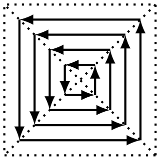
A toy problem on $\mathbb{T}^2$

Consider the problem of mixing $\mathbb{T}^2$ by a finite sequence of 90° rotations of squares. Given $x_0 \in \mathbb{T}^2$ and $r \in (0, 1/4)$, let $R_{x^0,r} : \mathbb{T}^2 \to \mathbb{T}^2$ be the map which rotates the square centered at x_0 of side length $2r$ by 90° counter-clockwise:

- Assign the *cost* r^2 to the rotation $R_{x,r}$.

Write $R_{x_0,r}(x) = \Phi(x, 1)$ where Φ is the incompressible flow generated for time $t \in [0, 1]$ by a (weakly) divergence free vector field $v = D_t\Phi$ such that

$$\|D_x v(\cdot, t)\|_{M(\mathbb{T})} = Cr^2 = C \cdot \text{cost of } R_{x_0,r}, \quad 0 \le t \le 1.$$



Discrete toy version of Bressan's conjecture

Now consider compositions

$$\Phi(\cdot, n) := R_{x_1,r_1} \circ \cdots \circ R_{x_n,r_n}$$

which are generated by a vector field $v(x,t)$, $t \in [0,n]$ with cost

$$\int_0^n \|D_x D_t \Phi(\cdot,t)\|_{M(\mathbb{T}^2)} dt = C \sum_{i=1}^n r_i^2.$$

Theorem

If $R_{x_1,r_1} \circ \cdots \circ R_{x_n,r_n}(0,\frac{1}{2})^2$ is mixed at scale $\varepsilon \in (0,1/2)$, then

$$\sum_{i=1}^n r_i^2 \geq C^{-1} \log \varepsilon^{-1}, \tag{1}$$

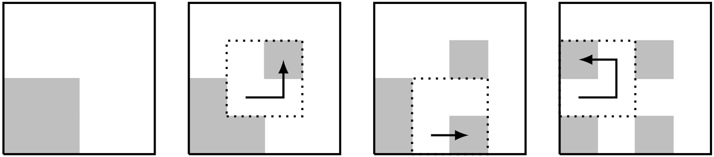
with a universal constant $C > 0$.

Sharpness of the toy theorem

To see the sharpness of the result consider the composition

$$R^3_{(\frac{r}{4},\frac{r}{2}),\frac{r}{4}} \circ R_{(\frac{r}{2},\frac{r}{4}),\frac{r}{4}} \circ R^2_{(\frac{r}{2},\frac{r}{2}),\frac{r}{4}}$$

which divides $(0, r/2)^2$ into four smaller squares, at cost $\sim 6r^2$:



Applying this idea recursively, we see that we can achieve mixing on the square $(0, r)^2$ to scale $2^{-n}r$ at cost Cnr^2.

III. Measuring mixing in the Bianchini seminorm

- One approach to Bressan's problem originated in a paper by S. Bianchini on a 1D analogue. Let

$$\mathrm{osc}_r(f,x) = \left| f(x) - \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy \right|$$

Note that if A is mixed at scale ε then

$$\mathrm{osc}_r(\mathbb{1}_A, x) \geq c, \quad x \in \mathbb{T}^d, \quad C\varepsilon \leq r \leq 1\,.$$

- Define the Bianchini semi-norm

$$\|f\|_{\mathcal{B}} = \int_0^{1/4} \left\| \mathrm{osc}_r(f,\cdot) \right\|_{L^1(\mathbb{T}^d)} \frac{dr}{r}$$

Thus: If A is mixed at scale ε then

$$\|\mathbb{1}_A\|_{\mathcal{B}} \gtrsim \log(1/\varepsilon).$$

Improved Crippa-DeLellis result via Bianchini norms

- For the toy result estimate $\|\mathbb{1}_{\Phi_k(E)}\|_{\mathcal{B}} - \|\mathbb{1}_E\|_{\mathcal{B}}$, for $k = 1, 2, \ldots$.

In the general case we prove:

Theorem

Let (Φ_t) be the incompressible flow on $\mathbb{T}^d$ associated with v, div $v = 0$.
Then for any measurable $E \subset \mathbb{T}^d$,

$$\|\mathbb{1}_{\Phi_T E}\|_{\mathcal{B}} \leq \|\mathbb{1}_E\|_{\mathcal{B}} + C \int_0^T \|Dv(\cdot, t)\|_{h^1} \, dt$$

h^1 is the local Hardy space. $L \log L \subset h^1$.

Minor technical details: Put $\|f\|_{\mathcal{B}(\delta)} := \int_\delta^{1/4} \|\mathrm{osc}_r(f,\cdot)\|_1 \frac{dr}{r}$ and estimate

$$\sup_{\delta<1/4} \left\{ \|\mathbb{1}_{\Phi_T E}\|_{\mathcal{B}(\delta)} - \|\mathbb{1}_E\|_{\mathcal{B}(\delta)} \right\}.$$

Given E define $f_E(x) = \mathbb{1}_E(x) - \mathbb{1}_{E^\complement}(x)$. Then $2\|\mathbb{1}_E\|_{\mathcal{B}(\delta)} = \|f_E\|_{\mathcal{B}(\delta)}$.

Proposition

Let v, Φ be as above. Then

$$\|\mathbb{1}_{\phi_T(E)}\|_{\mathcal{B}(\delta)} - \|\mathbb{1}_E\|_{\mathcal{B}(\delta)} =$$

$$\frac{1}{2V_d} \int_0^T \iint\limits_{\substack{|x-y|\in \\ (\delta,1/4)}} f_{\Phi_t(E)}(x) f_{\Phi_t(E)}(y) \frac{\langle x-y, v(x,t)-v(y,t)\rangle}{|x-y|^{d+2}} dxdy\, dt$$

$$+\ Error$$

Pf. of Prop. Use changes of variables (incompressibility)

$$\int \operatorname{osc}_r(f_E \circ \Phi_T^{-1}, x)dx - \int \operatorname{osc}_r(f_E, x)dx$$
$$= \int_{\mathbb{T}} \Big[\big| f_E(x) - \mathrm{Av}_{\phi_T^{-1} B_r(\phi_T(x))} f_E \big| - \big| f_E(x) - \mathrm{Av}_{B_r(x)} f_E \big| \Big] \, dx$$
$$= \int f_E(x) \Big[\fint_{B_r(x)} f_E(y)dy - \fint_{\phi_T^{-1} B_r(\phi_T(x))} f_E(y)dy \Big] \, dx$$
$$= -\int_0^T \int f_E(x) \, \frac{d}{dt} \Big[\fint_{\phi_t^{-1} B_r(\phi_t(x))} f_E(y)dy \Big] \, dx \, dt$$

Integrate in r

$$\int_\delta^{1/4} \fint_{\phi_t^{-1} B_r(\phi_t(x))} f_E(y)dy \, \frac{dr}{r} = \int H_\delta(\phi_t(x) - \phi_t(y)) f_E(y) \, dy$$

where H_δ is a singular kernel such that

$$\nabla H_\delta(u) = -V_d^{-1} \, \frac{u}{|u|^{d+2}} \, \chi_{\mathcal{A}(\delta,1/4)}(u).$$

We get

$$\frac{d}{dt}\Big[\int_\delta^{1/4}\int\int_{\phi_t^{-1}B_r(\phi_t(x))} f_E(y)dy\, f_E(x)dx\,\frac{dr}{r}\Big] =$$

$$\iint\limits_{\substack{|\phi_t(x)-\phi_t(y)\\ \in(\delta,1/4)}} f_E(y)f_E(x)\frac{\langle v(\phi_t(x),t)-v(\phi_t(y),t),\phi_t(x)-\phi_t(y)\rangle}{V_d|\phi_t(x)-\phi_t(y)|^{d+2}}\,dy\,dx.$$

and then after integrating in t and changing variables

$$\|f_{\phi_T(E)}\|_{\mathcal{B}(\delta)}-\|f_E\|_{\mathcal{B}(\delta)} =$$

$$\int_0^T\iint\limits_{\substack{(x,y):\\ \delta\le|x-y|\le\frac{1}{4}}} f_{\phi_t(E)}(x)f_{\phi_t(E)}(y)\frac{\langle v(x,t)-v(y,t),x-y\rangle}{V_d|x-y|^{d+2}}\,dy\,dx\,dt.$$

Flavien Léger's work (arXiv 2016):

Let $v(x,t)$ be as above, $\mathrm{div}_x v = 0$ and let $\theta(x,t)$ satisfy

$$\frac{\partial \theta}{\partial t} + \mathrm{div}(v\theta) = 0.$$

Consider

$$\mathcal{V}(f) = \int |\log|\xi|| \, |\widehat{f}(\xi)|^2 d\xi \; =$$

$$\alpha_d\Big(\frac{1}{2}\iint_{|h|\le 1} \frac{|f(x+h)-f(x)|^2}{|h|^d} dh\,dx - \iint_{|x-y|\ge 1} \frac{f(x)f(y)}{|x-y|^d} dxdy\Big) - \beta_d \|f\|_2^2$$

Then

$$\frac{d}{dt}\mathcal{V}(\theta(\cdot,t)) = c_d \iint \theta(x,t)\theta(y,t) \frac{\langle v(x,t)-v(y,t), x-y\rangle}{|x-y|^{d+2}} \, dy\,dx.$$

IV. The Singular Integral Kernel

We need to consider, for a vector field b with $\operatorname{div} \vec{b} = 0$

$$\frac{\langle b(x) - b(y), x - y \rangle}{|x - y|^{d+2}} = \sum_{(i,j)\neq(d,d)} K_{ij}(x-y) \int_0^1 a_{ij}(sx+(1-s)y)ds$$

with the *even* Calderón-Zygmund convolution kernels

$$K_{ij}(w) = \frac{w_i w_j}{|w|^{d+2}}, \qquad a_{ij} = \frac{\partial b_i}{\partial x_j}, \qquad i \neq j$$

$$K_{ii}(w) = \frac{w_i^2 - w_d^2}{|w|^{d+2}}, \qquad a_{ii} = \frac{\partial b_i}{\partial x_i}, \qquad i \leq d-1$$

- Look like instances of Christ-Journé kernels ("first order d-commutators"). These are rough higher dimensional variants of the Calderón commutators with $a \equiv a_{ij} \in L^\infty$. Christ-Journé (1987) proved L^p estimates with $a \in L^\infty$.
- We have $a \in L^p$ or even $a \in L \log L$.

Weak Type (1,1) Bounds for CJ-operators, $d \geq 2$

$$\mathcal{T}[f,a](x) := \int K(x-y) \int_0^1 a(sx+(1-s)y)ds\, f(y)dy.$$

Theorem (Grafakos-Honzík ($d = 2$), and A.S. ($d \geq 2$))

$$\text{meas}(\{x : |\mathcal{T}[f,a](x)| > \lambda\}) \lesssim \frac{\|a\|_\infty \|f\|_1}{\lambda}.$$

• Shares some features with rough convolution SIO's (Christ-Rubio de Francia 88, Hofmann 88, A.S. 96, Tao 99). One also has:

Theorem

$$\text{meas}(\{x : |\mathcal{T}[f,a](x)| > \lambda\}) \lesssim \frac{\|a\|_1 \|f\|_\infty}{\lambda}.$$

Yields the $L \log L$ bound in the previous theorem.

[HSSS]

Note that the weak type (1,1) inequalities for the Christ-Journé operators cannot be replaced by $H^1 \to L^1$ estimates (H^1 =Hardy space). But we do have

Theorem

Suppose $div(b) = 0$. *Let*

$$T[f, b](x) = \int \frac{\langle b(x) - b(y), x - y \rangle}{|x - y|^{d+2}} f(y) dy$$

Then

$$\|T[f, b]\|_{L^1} \lesssim \|f\|_\infty \|\nabla b\|_{H^1} .$$

arXiv:1612.03431

V. Open problems I: Smoothness zero Besov spaces

• Given any of the numerous characterizations of the Besov-Nikolskij-Taibleson spaces $B^s_{p,q}$ for $s > 0$ there are limiting cases for $s = 0$ which define new spaces.
These limiting spaces do not usually coincide with the Fourier analytically defined Besov space $B^0_{p,q}$.

An active area in the theory of function spaces is concerned with the question:

Q. What is the precise interrelation between all these spaces (including the Bianchini space)?

Cf. various results and references in the talk by Óscar Domínguez.

Open problems, II: related to Bressan's question

- Bressan's problem for suitable spaces X between $L \log L$ and L^1.

- Specifically, if $b \in W_1^1(\mathbb{T}^d)$ with $\mathrm{div}(b) = 0$, can we say something meaningful about

$$\iint_{\substack{(x,y):\\ \delta \leq |x-y| \leq \frac{1}{4}}} f_E(x) f_F(y) \frac{\langle b(x) - b(y), x - y\rangle}{|x-y|^{d+2}}\, dy\, dx$$

for $F = E$ or $F = E^{\complement}$, and $f_E := \mathbb{1}_E - \mathbb{1}_{E^{\complement}}$?

Remark: There are counterexamples for just slightly more general E, F.

Open problems, III: Rough singular integrals

- Estimates for Christ-Journé biinear operators with target space L^p, for some $p \leq 1$?

  $$\mathcal{T}[f_1, f_2](x) = \int K(x-y) \int_0^1 f_2(sx + (1-s)y) ds\, f_1(y) dy.$$

  Known [CJ, SSS]:
  $\mathcal{T} : L^{p_1} \times L^{p_2} \to L^p$, $\frac{p}{p_1} + \frac{p}{p_2} = 1$ for $p_1, p_2 > 1$ and $p > 1$.
  What about $p > \frac{d}{d+1}$?

- Are there theorems for almost everywhere existence of the p.v. version of *rough* singular integrals (convolution, or Christ-Journé), when $f \in L^1$?