Classical properties of composition operators on Hardy–Orlicz spaces on planar domains

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Composition operators on spaces of analytic functions

Let $H(\Omega)$ denote the space of all holomorphic functions on Ω, where Ω is a domain on the Riemann sphere. For analytic map $\varphi \in \Upsilon_\Omega := \{f \in H(\Omega) \colon f(\Omega) \subset \Omega\}$ the composition operator is defined by

$$C_\varphi f := f \circ \varphi, \quad f \in H(\Omega).$$

Problem. Relate operator theoretic properties (e.g., boundedness, compactness, weak compactness, order boundedness, spectral properties) of composition operator $C_\varphi \colon X(\Omega) \to X(\Omega)$ to function theoretic properties of generating function φ (symbol of C_φ).

Hardy-type spaces

- For $0 < p < \infty$, $\Omega = \mathbb{D} := \{z \in \mathbb{C}\,;\, |z| < 1\}$,

$$H^p := H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D})\,;\, \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < +\infty \right\}$$

- For Orlicz function Φ (i.e. $\Phi\colon [0, +\infty) \to [0, +\infty)$ is convex and nondecreasing function with $\Phi(0) = 0, \lim_{x \to +\infty} \Phi(x) = +\infty$),

$$H^\Phi := H^\Phi(\mathbb{D}) = \{f \in H(\mathbb{D})\,;\, \|f\|_{H^\Phi(\mathbb{D})} < +\infty\},$$

where $\|f\|_{H^\Phi(\mathbb{D})} := \sup_{0 \le r < 1} \|f_r\|_\Phi = \lim_{r \to 1^-} \|f_r\|_\Phi$,
$f_r(t) := f(re^{it})$,

$$\|f_r\|_\Phi = \inf \left\{ \varepsilon > 0\,;\, \int_0^{2\pi} \Phi\Big(\frac{|f(re^{it})|}{\varepsilon} \Big)\, dt \le 1 \right\}.$$

Composition operators on Hardy spaces

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- [B. D. MacCluer, 1985] $C_\varphi \colon H^p \to H^p$ is compact if and only if the pullback measure μ_φ defined by formula

$$\mu_\varphi(B) = m((\varphi^*)^{-1}(B)),$$

  where B is a Borel subset of $\mathbb{D}$ and m is normalized Lebesgue measure on $\partial\mathbb{D}$ is vanishing Carleson measure i.e., $\mu_\varphi(W(a,h)) = o(h)$ as $h \to 0$ for any Carleson window $W(a,h) = \{z \in \mathbb{D} \colon 1-h < |z| < 1, |\arg(\bar{a}z)| < h\}$, $a \in \partial\mathbb{D}$,

Composition operators on Hardy spaces

- [J. H. Shapiro, 1987] $C_\varphi \colon H^p \to H^p$ is compact if and only if,

$$\lim_{|w| \to 1^-} \frac{N_\varphi(w)}{-log|w|} = 0,$$

where N_φ is Nevanlinna counting function

$$N_\varphi(w) = \sum_{\varphi(z)=w} -\log|z|, \quad w \in \mathbb{D} \setminus \varphi(0).$$

Composition operators on Hardy–Orlicz spaces

- [P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, 2010]
  $C_\varphi \colon H^\Phi \to H^\Phi$ is compact if and only if,

$$\lim_{h \to 0^+} \frac{\Phi^{-1}\left(\frac{1}{h}\right)}{\Phi^{-1}\left(\frac{1}{\rho_\varphi(h)}\right)} = 0,$$

  where

$$\rho_\varphi(h) := \sup_{\xi \in \partial\mathbb{D}} \mu_\varphi(W(\xi, h)),$$

  is called *Carleson function*.

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- $\Phi \in \Delta^2 \iff \exists \alpha > 1 \quad \exists t_0 > 0 \quad \forall t \geq t_0 \quad \Phi(\alpha t) \geq [\Phi(t)]^2$

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Theorem

Let $\varphi \in \Upsilon_{\mathbb{D}}$ and suppose that $\Phi \in \Delta^2$. The following assertions are equivalent:

(1) $C_\varphi \colon H^\Phi \to H^\Phi$ *is compact,*
(2) $C_\varphi \colon H^\Phi \to H^\Phi$ *is weakly compact,*
(1) $C_\varphi \colon H^\Phi \to H^\Phi$ *is order bounded in* $M^\Phi(\partial\mathbb{D})$.

Hardy spaces on general domains

Problem. How to define $H^p(\Omega)$ for general Ω?

Definition. (W. Rudin, 1955) A function $f \in H(\Omega)$ belongs to $H^p(\Omega)$ if and only if the subharmonic function $|f|^p$ has harmonic majorant v. Moreover

$$\|f\|_{H^p(\Omega)} = v_f(z_0)^{1/p},$$

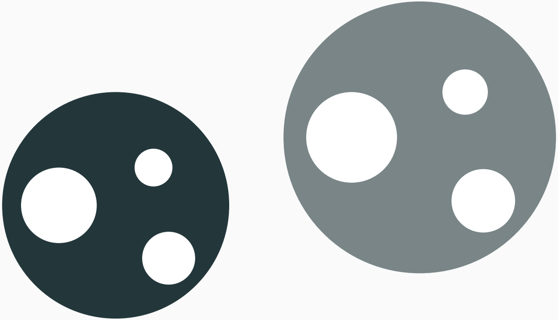
where v_f is the least harmonic majorant $|f|^p$ for a fixed $z_0 \in \Omega$.

Circular domains

A domain Ω is called *a circular domain* if

$$\Omega = \mathbb{D} \setminus \bigcup_{i=1}^{m} (a_i + r_i \overline{\mathbb{D}})$$

where a_i's belong to $\mathbb{D}$ and $r_i \in (0, 1)$ and the circumferences $\partial\mathbb{D}, a_1 + r_1\partial\mathbb{D}, \ldots, a_m + r_m\partial\mathbb{D}$ do not intersect. Denote by E_i the following sets: $E_i = \mathbb{C}_\infty \setminus (a_i + r_i\overline{\mathbb{D}})$ for $i = 1, \ldots, m$ and $E_0 = \mathbb{D}$.



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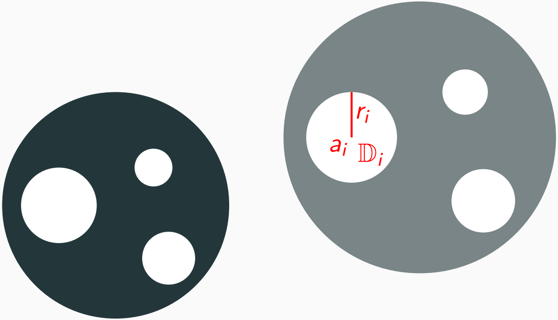

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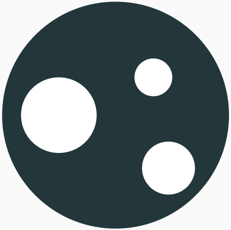

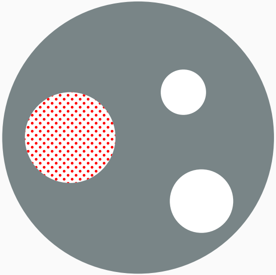

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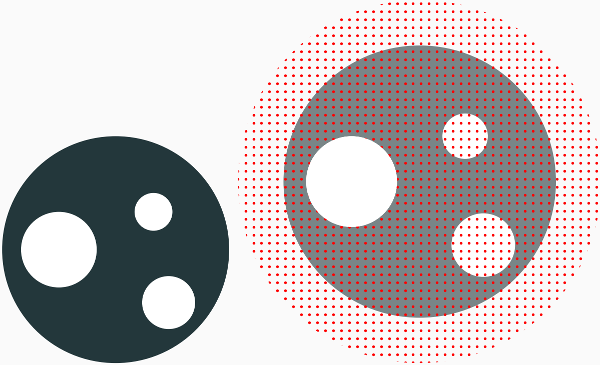

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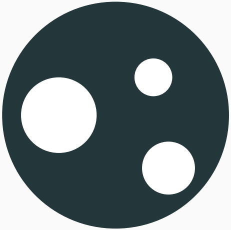
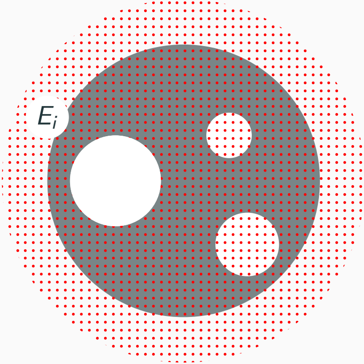




Definition. *Regular exhaustion* of Ω is a sequence $\{\Omega_n\}_{n=1}^{\infty}$ of subdomains of Ω which satisfy the following conditions:

(1) $\bar{\Omega}_n \subset \Omega_{n+1}$ for $n \in \mathbb{N}$,

(2) $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$,

(3) every component of $\partial\Omega_n$, is nontrivial for each $n \in \mathbb{N}$.

For fixed $z_0 \in \Omega_1$ denote by ω_{n,z_0} *the harmonic measure* on $\partial\Omega_n$. If $g\colon \Omega \to \mathbb{C}$ we write $g_n := g|_{\partial\Omega_n}$.

Definition.

$$H^{\Phi}(\Omega) := \{f \in H(\Omega)\,;\, \|f\|_{H^{\Phi}(\Omega)} < \infty\},$$

where $\|f\|_{H^{\Phi}(\Omega)} := \lim_{n\to\infty} \|f_n\|_{\Phi}$, $f_n := f|_{\partial\Omega_n}$,

$$\|f_n\|_{\Phi} = \inf\left\{\varepsilon > 0\,;\, \int_{\partial\Omega_n} \Phi\Big(\frac{|f_n|}{\varepsilon}\Big)\, d\omega_{n,z_0} \le 1\right\}.$$

Equivalently we can describe $H^{\Phi}(\Omega)$ in terms of harmonic majorant; it is a set of all holomorphic functions f, for which there exists $\lambda > 0$, such that subharmonic function $\Phi(\lambda|f|)$ has harmonic majorant. Moreover

$$\|f\|_{H^{\Phi}(\Omega)} = \inf\{\varepsilon > 0 : v_{f,\varepsilon(z_0)} \le 1\},$$

where $v_{f,\varepsilon}$ is the least harmonic majorant of $\Phi\big(\frac{|f|}{\varepsilon}\big)$.

We denote by $HM^{\Phi}(\Omega)$ the closure of $H^{\infty}(\Omega)$ in the norm of $H^{\Phi}(\Omega)$ and we denote by $H_0^{\Phi}(E_k)$ the subspace of $H^{\Phi}(E_k)$ which contains all such functions f that $f(\infty) = 0$.

Theorem

For each $f \in H^{\Phi}(\Omega)$ we have the following decomposition

$$f(z) = f_0(z) + f_1(z) + \dots + f_m(z), \quad z \in \Omega, \tag{1}$$

where $f_0 \in H^{\Phi}(E_0)$ and $f_k \in H_0^{\Phi}(E_k)$ for each $1 \leq k \leq m$. Moreover, the map $f \mapsto f_0$ is a bounded linear projection of $H^{\Phi}(\Omega)$ onto $H^{\Phi}(E_0)$ and $f \mapsto f_k$ is a bounded linear projection of $H^{\Phi}(\Omega)$ onto $H_0^{\Phi}(E_k)$.

Theorem

Every $f \in H^{\Phi}(\Omega)$ *has boundary values* f^* *almost everywhere* $(d\omega)$ on $\partial\Omega$ and $f^* \in L^{\Phi}(\Gamma, \omega)$. Moreover

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f^*(w)}{w - z}\, dw, \quad z \in \Omega, \tag{2}$$

$$0 = \int_{\partial\Omega} \frac{f^*(w)}{w - z} dw, \quad z \notin \bar{\Omega}, \tag{3}$$

$$f(z) = \int_{\partial\Omega} f^*(\zeta)\, d\omega_z(\zeta), \quad z \in \Omega. \tag{4}$$

Finally, the mapping $f \mapsto f^*$ is an isomorphism of $H^{\Phi}(\Omega)$ onto closed subspace of $L^{\Phi}(\partial\Omega, \omega)$ and it is an isometry of $HM^{\Phi}(\Omega)$ onto closed subspace of $L^{\Phi}(\partial\Omega, \omega)$.

Composition operators on $H^{\Phi}(\Omega)$

Denote the components of boundary of Ω as follow: $\partial\Omega = \Gamma_0 \cup \cdots \cup \Gamma_m$.
Definition. Carleson window of center $\xi \in \Gamma_i$ and radius $h \in (0, \min_{i \neq j} \operatorname{dist}(\Gamma_i, \Gamma_j))$ is the set

$$W_0(\xi, h) = \{z \in \Omega\,;\, 1 - h < |z|, |\arg(\bar{\xi} z)| < h\}, \quad i = 0$$

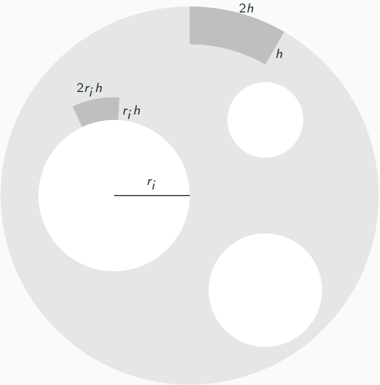
$$W_i(\xi, h) = \left\{z \in \Omega\,;\, |z - a_i| < \frac{r_i}{1 - h}, |\arg(\bar{\xi} z)| < h\right\}, \quad 1 \leq i \leq m.$$

- Let $\varphi\colon \Omega \to \Omega$ be holomorphic. Then for every Borel set $B \subset \Omega$, we define μ_φ by formula $\mu_\varphi(B) := \omega(\varphi^{*-1}(B))$. The function

$$\rho_{\mu_\varphi}(h) := \max_{0 \leq i \leq m} \sup_{\xi \in \Gamma_i} \mu_\varphi(W(\xi, h)),$$

  is called *Carleson function*.

Carleson windows



Carleson measures and compact composition operators

The Orlicz function Φ satisfies the ∇_2-condition ($\Phi \in \nabla_2$) if for some constant $\beta > 1$ and for some $t_0 > 0$, one has $\Phi(\beta t) \geq 2\beta\Phi(t)$, for $t \geq t_0$.

Theorem

Suppose that $\Phi \in \nabla_2$. Composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if

$$\lim_{h\to 0^+} \frac{\Phi^{-1}\left(\frac{1}{h}\right)}{\Phi^{-1}\left(\frac{1}{\rho_\varphi(h)}\right)} = 0.$$

Definition. For $a \in \partial\mathbb{D}$ and $r \in (0,1)$

$$u^0_{a,r}(z) := \left(\frac{1-r}{1-\bar{a}rz}\right)^2, \quad z \in \Omega,$$

and for $1 \leq i \leq m$ and a and r as above we define

$$u^i_{a,r}(z) = (u_{a,r} \circ \eta_i^{-1})(z) = \left(\frac{1-r}{1-\frac{\bar{a}rr_i}{z-a_i}}\right)^2, \quad z \in \Omega.$$

Compact composition operators

Theorem

Let $\varphi \in \Upsilon_\Omega$ and let $\Phi \in \nabla_2$ be an Orlicz function. The composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is compact if and only if, for each $0 \leq i \leq m$, we have

$$\lim_{s \to 1^-} \sup_{p \in \partial\mathbb{D}} \Phi^{-1}\left(\frac{1}{1-s}\right) \| C_\varphi u^i_{p,s} \|_{H^\Phi(\Omega)} = 0.$$

Order bounded composition operators

The operator $T\colon X \to Z$ from a Banach space X into a Banach subspace Z of a Banach lattice Y is order bounded if there is some positive $y \in Y$ such that $|Tx| \leq y$ for every x in the unit ball B_X of X.

Let Ω be a circular domain and $\varphi \in \Upsilon_\Omega$. The operator $\widetilde{C}_\varphi\colon H^\Phi(\Omega) \to L^\Phi(\partial\Omega, \omega)$ given by

$$\widetilde{C}_\varphi f = (C_\varphi f)^*$$

is well defined.

Theorem

Let Φ be an Orlicz function and $\varphi \in \Upsilon_\Omega$. The composition operator $C_\varphi\colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ induces an operator $\widetilde{C}_\varphi\colon H^\Phi(\Omega) \to L^\Phi(\partial\Omega)$ given by $\widetilde{C}_\varphi f = (C_\varphi f)^*$ which is order bounded in $L^\Phi(\partial\Omega)$ (respectively in $M^\Phi(\partial\Omega)$) if and only if $\operatorname{dist}(\varphi^*, \partial\Omega) > 0$ ω-a.e. and the function $\Phi^{-1}\left(\frac{1}{\operatorname{dist}(\varphi^*, \partial\Omega)}\right)$ belongs to $L^\Phi(\partial\Omega)$ (resp. belongs to $M^\Phi(\partial\Omega)$).

Order bounded composition operators

Theorem

Let Ω be a circular domain, $\varphi \in \Upsilon_\Omega$, and let Φ be an Orlicz function such that $\Phi \in \Delta^2$. The following assertions are equivalent:

(1) $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact.*

(2) $\widetilde{C_\varphi} \colon H^\Phi(\Omega) \to M^\Phi(\partial\Omega)$ *is order bounded.*

Weakly compact composition operators

The Orlicz function Φ satisfies the Δ^0-condition ($\Phi \in \Delta^0$), if there exists $\beta > 1$ such that

$$\lim_{x \to +\infty} \frac{\Phi(\beta x)}{x} = +\infty.$$

Theorem

Let Ω be a circular domain and $\varphi \in \Upsilon_\Omega$. Assume that the Orlicz function Φ satisfies the Δ^0-condition. If the composition operator $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ is weakly compact, then for each $i \in \{0, \ldots, m\}$ the following condition is satisfied

$$\lim_{s \to 1^-} \sup_{p \in \partial\mathbb{D}} \Phi^{-1}\Big(\frac{1}{1-s}\Big) \| C_\varphi u^i_{p,s} \|_{H^\Phi(\Omega)} = 0.$$

Weakly compact composition operators

$$\Phi \in \Delta^2 \implies \Phi \in \Delta^0 \implies \Phi \in \nabla_2$$

Theorem

Let Ω be a circular domain, $\varphi \in \Upsilon_\Omega$. Assume that Φ is an Orlicz function and $\Phi \in \Delta^2$. The following assertions are equivalent:

(1) $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact.*

(2) $\widetilde{C_\varphi} \colon H^\Phi(\Omega) \to M^\Phi(\partial\Omega)$ *is order bounded.*

(3) $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is weakly compact.*

(4) *For each $i \in \{0, \dots, m\}$ we have*

$$\lim_{s \to 1^-} \sup_{p \in \partial\mathbb{D}} \Phi^{-1}\left(\frac{1}{1-s}\right) \| C_\varphi u^i_{p,s} \|_{H^\Phi(\Omega)} = 0.$$

Connections between compactness and weak compactness on $H^2(\Omega)$ and $H^\Phi(\Omega)$

Theorem

Let Ω be a circular domain, $\varphi \in \Upsilon_\Omega$. Assume that Φ is an Orlicz function and $\Phi \in \nabla_2$. If one of the following conditions:

(1) $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is compact,*

(2) $\Phi \in \Delta^0$ *and* $C_\varphi \colon H^\Phi(\Omega) \to H^\Phi(\Omega)$ *is weakly compact,*

is satisfied, then the composition operator $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ is compact.

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