Pseudo-differential operators on metric measure spaces

Yauhen Radyna

Mechanics and Mathematics Faculty
Belarusian State University

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The Aim

The aim of the talk is to discuss a construction of operator algebra acting in function space over metric measure space. In case of Euclidean space this should produce a (sub)algebra of pseudo-differential operators.

We replace in general non-existent local differentiation operators with linear operators which use Steklov averaging in its construction.

Current limitations: we consider s-regular metric measure spaces.

Content

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Consider a family $M = (M_t)_{0<t<+\infty}$ of uniformly bounded operators acting in $L^q(X)$ such that

1. $M_t\psi \to \psi$ in $L^q(X)$ and a.e. as $t \to 0+$;
2. $M_t\psi \to 0$ in $L^q(X)$ and a.e. as $t \to +\infty$;
3. $M_t\psi$ is of bounded variation w.r.t. $t \in (0, +\infty)$ for a.a. x.

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Each M_t performs "averaging at scale t".
We refer to M as "the family of averaging operators".

Family of averaging operators M_t. Example 1

Steklov mean operators provide the main example:

$$(M_t\psi)(x) = \psi_{B[x,t]} = \frac{1}{\mu B[x,t]} \int_{B[x,t]} \psi \, d\mu.$$

Family of averaging operators M_t. Example 2

Let A be a dissipative operator in $L^q(X)$, $q > 1$,

$$\forall \lambda > 0 \ \forall \psi \in Dom(A): \ ||(\lambda I - A)\psi|| \geq \lambda ||\psi||.$$

Consider an operator semi-group

$$M_t \psi = \exp(tA)\psi, \quad M_{t+s} = M_t \circ M_s, \ t, s > 0.$$

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What are sufficient conditions on A for averaging operators axioms to hold? In practice, axioms often hold, e.g. for $A = \Delta$, the Laplacian in $L^q(\mathbb{R}^n)$.

M. Fukushima, Y. Oshima, M. Takeda. Dirichlet Forms and Symmetric Markov Processes. 2nd ed. de Gruyter, Berlin, 2011.

Operator $D[h]$ in $L^q(X)$, $q > 1$

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$$\psi(x) = -\int_0^{+\infty} 1 \, d(M_t\psi(x)).$$

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Let $h : (0, +\infty)$ be a Borel function. Consider operator $D[h]$ in $L^q(X)$

$$(D[h]\psi)(x) = -\int_0^{+\infty} h(t) d(M_t\psi(x)).$$

Operator $D[h]$ in $L^q(\mathbb{R}^n)$. Example

Pseudo-differential operator $D_\alpha = |\Delta|^{\alpha/2}$ defined for ϕ in Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$|\Delta|^{\alpha/2}\psi = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}\psi$$

can be regarded as a $D[h]$ if one takes Steklov mean operators as M and $t \mapsto c_\alpha t^{-\alpha}$ as h:

$$|\Delta|^{\alpha/2}\psi(x) = c_\alpha \int_0^{+\infty} t^{-\alpha} d(M_t\psi(x)), \quad M_t\psi(x) = \psi_{B[x,t]}.$$

Operators $D[h]$ in $L^q(\mathbb{R}^n)$. Properties

All off $D[h]$'s are convolution operators in $L^q(\mathbb{R}^n)$. They commute:

$$D[f] \circ D[g]\ \psi = D[g] \circ D[h]\ \psi$$

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Composition of $D[h]$'s gives rise to a composition of "symbols":

$$D[f] \circ D[g] = D[f * g],$$

where $f * g$ stands for commutative and associative convolution of functions on $(0, +\infty)$.

Operator $D[h]$ in $L^q(X)$. Example

Analogue of $D_\alpha = |\Delta|^{\alpha/2}$ can be defined in $L^q(X)$:

$$D_\alpha\psi(x) = \int_0^{+\infty} t^{-\alpha} d(M_t\psi(x)), \quad M_t\psi(x) = \psi_{B[x,t]}.$$

Operators $D[h]$ in $L^q(X)$. Properties

$D[h]$'s do not commute in general:

$$D[f] \circ D[g] \neq D[g] \circ D[h].$$

But

$$D_\alpha \circ D_\beta - D_\beta \circ D_\alpha \text{ is bounded,}$$
$$D_\alpha \circ D_\beta - D_{\alpha+\beta} \text{ is bounded.}$$

Sobolev spaces $W^{q,\beta}(X)$

Definition is straightforward:

$$W^{q,\beta}(X) = \{\psi \in L^q(X) : D_\beta \psi \in L^q(X)\}, \quad q > 1, \beta > 0.$$

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In case of $X = \mathbb{R}^n$ the definition gives the usual Sobolev spaces

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The definition gives the fractional Hajlasz–Sobolev class $M^{q,\beta}$ when $0 < \beta \leq 1$:

$$M^{q,\beta} = \{\psi \in L^q(X) : \exists g \in L^q(X) : |\psi(x) - \psi(y)| \leq d(x,y)^\beta [g(x) + g(y)] \text{ a.e.}\}$$

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Some expected inclusions are true:

$$W^{q,\beta}(X) \supset W^{q,\beta^*}(X) \text{ as } \beta \le \beta^*.$$

Pseudodifferentail operators. General idea

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- metric measure space with doubling condition;

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- metric measure space with doubling condition;

LCA groups approach:

Karlheinz Gröchenig and Thomas Strohmer. Pseudodifferential Operators on Locally Compact Abelian Groups and Sjöstrand's Symbol Class, *J. Reine Angew. Math.* (2006)

Radial functions

The Fourier transform of radial functions are radial functions.
In definition of pseudodifferential operator

$$P(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)\, d\xi$$

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Radial function approach:

Frédéric Bernicot, Dorothee Frey. Pseudodifferential Operators Associated with a Semigroup of Operators *J. Fourier Anal. Appl.* (2014)

We consider operators $P(x,|D|)$ rather than $P(x,D)$.

Symbol Classes

It is more advantageous to use not a Steklov averaging family but some heat semigroup associated to analogue of Laplacian (we consider it also to be of order 2, it's not necessary however).

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For $\rho,\delta \in [0,1]$ and $s>0$ we say that symbols $\sigma(x,\xi) \in C^\infty(X \times (0,+\infty))$ falls in the class $S^{s,m}_{\rho,\delta}(\Delta)$ if it satisfies

$$\forall \alpha,\beta > 0, \quad |\partial_\xi^\beta \Delta^\alpha \sigma(x,\xi)| \leq C(1+|\xi|)^{\frac{s}{m}-\rho\beta+\frac{2}{m}\delta\alpha}.$$

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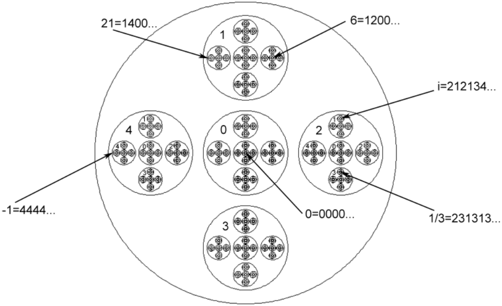
The construction of the operator is done first for

$$\sigma(x,\xi)=\sigma_1(x)\sigma_2(\xi)$$

for smooth $\sigma_1(x)$ and smooth compactly supported functions $\sigma_2(\xi)$.

p-Adic Pseudodifferential Operators



Pic. Unit ball in p-adic field $\mathbb{Q}_5$

Vladimirov V.S., Volovich I.V., Zelenov E.I. (1989), Haran S. (1993) considered classes of pseudodifferential operators on $\mathbb{Q}_p$.

Concept of a direction

What is the direction in metric space?

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Can it be derived using an embedding of metric measure space into euclidean spaces?

Thank you for your attention!