Besov and Triebel-Lizorkin spaces associated with Laguerre expansions of Hermite type

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Bedlewo, 18.09.2017

Agenda

Recent articles

- P. Petrushev, Y. Xu, *Decoposition of spaces of distributions induced by Hermite expansions*, J. Fourier Anal. Appl. (2008)
- T.A. Bui, X.T. Duong, *Besov and Triebel-Lizorkin Spaces Associated to Hermite Operators*, J. Fourier Anal. Appl. (2015)
- T. A. Bui, X. T. Duong, *Laguerre operator and its associated weighted Besov and Triebel–Lizorkin spaces*, Trans. Amer. Math. Soc.(2017)
- P. Plewa, *Besov and Triebel-Lizorkin spaces associated with Laguerre expansions of Hermite type*, Acta Math. Hung. (2017)

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Laguerre functions

The **Laguerre functions of Hermite type** of order $\alpha > -1$ on $\mathbb{R}_+$ are the functions

$$\varphi_k^\alpha(x) = \Big(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\Big)^{1/2} L_k^\alpha(x^2) x^{\alpha+1/2} e^{-x^2/2}, \qquad x > 0,$$

where L_k^α denotes the Laguerre polynomial of degree k and order α defined by

$$L_k^\alpha(x) = \frac{x^{-\alpha} e^x}{k!} \frac{d^k}{dx^k}\Big(e^{-x} x^{k+\alpha}\Big).$$

In the multidimensional case we take the tensor product.
The functions $\{\varphi_k^\alpha : \ k \in \mathbb{N}^d\}$ form an orthonormal basis in $L^2(\mathbb{R}_+^d, dx)$.

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Laguerre operator

Consider the operator

$$L_\alpha = -\Delta + |x|^2 + \sum_{i=1}^{d} \frac{\alpha_i^2 - 1/4}{x_i^2}.$$

The functions φ_k^α are its eigenfuctions corresponding to the eigenvalue $\lambda_{|k|}^\alpha = 4|k| + 2|\alpha| + 2d$.

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Operators $P^{\alpha}_{t,m}$

For $m \in \mathbb{N}$ we consider the family of operators

$$P^{\alpha}_{t,m} = \left(t\sqrt{L_\alpha}\right)^m e^{-t\sqrt{L_\alpha}}.$$

The operators $P^{\alpha}_{t,m}$ are spectrally defined on $L^2(\mathbb{R}^d_+)$ by

$$P^{\alpha}_{t,m}\phi = \sum_{n=0}^{\infty} \left(t\sqrt{\lambda^{\alpha}_n}\right)^m e^{-t\sqrt{\lambda^{\alpha}_n}} \sum_{|k|=n} \langle \phi, \varphi^{\alpha}_k \rangle \varphi^{\alpha}_k.$$

Moreover, the operators $P^{\alpha}_{t,m}$ are integral operators:

$$\forall \phi \in L^2(\mathbb{R}^d_+) \qquad P^{\alpha}_{t,m}\phi(x) = \int_{\mathbb{R}^d_+} p^{\alpha}_{t,m}(x,y)\phi(y)dy, \qquad x \in \mathbb{R}^d_+,\ t > 0.$$

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Kernels $p^{\alpha}_{t,m}$

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$$p^{\alpha}_{t,m}(x,y) = \sum_{n=0}^{\infty} \left(t\sqrt{\lambda^{\alpha}_n}\right)^m e^{-t\sqrt{\lambda^{\alpha}_n}} \sum_{k=|n|} \varphi^{\alpha}_k(x)\varphi^{\alpha}_k(y), \qquad x,y \in \mathbb{R}^d_+.$$

Lemma

For every $m \in \mathbb{N}_+$ and $\alpha \in [-1/2,\infty)^d$ we have

$$|p^{\alpha}_{t,m}(x,y)| \lesssim \frac{t^m}{(t+|x-y|)^{d+m}}, \qquad t>0, \quad x,y \in \mathbb{R}^d_+.$$

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Space of test functions

Theorem

Let $\phi \in L^2(\mathbb{R}^d)$. Then $\phi \in \mathcal{S}(\mathbb{R}^d)$ if and only if for every $N \in \mathbb{N}$

$$\langle \phi, h_k \rangle_{L^2(\mathbb{R}^d, dx)} = O((1 + |k|)^{-N}),$$

uniformly in $k \in \mathbb{N}^d$.

Natural definition:

$$\mathcal{S}_\alpha = \Big\{ \phi \in L^2(\mathbb{R}^d_+, dx) : \quad \forall N \in \mathbb{N}$$
$$\langle \phi, \varphi^\alpha_k \rangle_{L^2(\mathbb{R}^d_+, dx)} = O\Big((1 + |k|)^{-N} \Big), \quad k \in \mathbb{N}^d \Big\}.$$

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Characterization of $\mathcal{S}_\alpha$

A function f defined on $\mathbb{R}^d$ is called multi-even if

$$f(x_1, \dots, x_d) = f(|x_1|, \dots, |x_d|), \qquad (x_1, \dots, x_d) \in \mathbb{R}^d.$$

We define Schwartz space $\mathcal{S}_e$ on $\mathbb{R}^d_+$, as the space of restrictions of multi-even functions from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{R}^d_+$.

Theorem

If $\alpha \in (-1, \infty)^d$, then

$$\mathcal{S}_\alpha = x^{\alpha+1/2} \cdot \mathcal{S}_e.$$

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Definitions of the spaces

For $\sigma \in \mathbb{R}$, $0 < p, q \leqslant \infty$ and $m \in \mathbb{N}$, such that $m > m_0$ we define the *homogeneous Besov spaces* $\dot{B}^{\sigma, L_\alpha, m}_{p,q}$ by

$$\dot{B}^{\sigma, L_\alpha, m}_{p,q} =$$

$$\left\{ f \in \mathcal{S}'_\alpha : \ \|f\|_{\dot{B}^{\sigma, L_\alpha, m}_{p,q}} = \Big(\int_0^\infty \left(t^{-\sigma} \left\| P^\alpha_{t,m} f \right\|_p \right)^q \frac{dt}{t} \Big)^{1/q} < \infty \right\}$$

and if $p < \infty$ we define the *homogeneous Triebel-Lizorkin spaces* $\dot{F}^{\sigma, L_\alpha, m}_{p,q}$ by

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Molecules

Let $0 < p \leqslant \infty$, $\sigma \in \mathbb{R}$, and $M, N \in \mathbb{N}_+$. A function $a \in L^2(\mathbb{R}^d_+)$ is called $(L_\alpha, M, N, \sigma, p)$ *molecule* if there exist a function b and a dyadic cube $Q \in \mathcal{D}_\nu$, such that

(i) $\left(\sqrt{L_\alpha}\right)^M b = a$;

(ii) $\left|\left(\sqrt{L_\alpha}\right)^j b(x)\right| \lesssim 2^{\nu(M-j+\sigma)}|Q|^{-1/p}\left(1+\frac{|x-x_Q|}{2^\nu}\right)^{-d-N}$ for $j = 0, \ldots, 2M$ and $x \in \mathbb{R}^d_+$ a.e.

Theorems

From now on let $\alpha \in [-1/2, \infty)^d \setminus (-1/2, 1/2)^d$.

Theorem 1

Let $\sigma \in \mathbb{R}$, $0 < p, q \leqslant \infty$, $m_1, m_2 \in \mathbb{N}$ and $m_1, m_2 > m_0$. Then the spaces $\dot{B}^{\sigma, L_\alpha, m_1}_{p,q}$ and $\dot{B}^{\sigma, L_\alpha, m_2}_{p,q}$ coincide and their norms are equivalent.

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Theorems

Theorem 2

Let $\sigma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. For $M, N \in \mathbb{N}_+$ and $m \in \mathbb{N}$, such that $m > m_0$, if $f \in \dot{B}^{\sigma, L_\alpha, m}_{p,q}$, then there exist a sequence of $(L_\alpha, M, N, \sigma, p)$ molecules $\{a_Q\}_{Q \in \mathcal{D}}$ and a sequence of coefficients $\{s_Q\}_{Q \in \mathcal{D}}$ such that

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q$$

in $\mathcal{S}'_\alpha$.

Moreover,

$$\Big[\sum_{\nu \in \mathbb{Z}} \Big(\sum_{Q \in \mathcal{D}_\nu} |s_Q|^p \Big)^{q/p} \Big]^{1/q} \simeq \|f\|_{\dot{B}^{\sigma, L_\alpha, m}_{p,q}}.$$

Conversely, if $M, N, m \in \mathbb{N}$ are sufficiently large and for a sequence of $(L_\alpha, M, N, \sigma, p)$ molecules $\{a_Q\}_{Q\in\mathcal{D}}$ and a sequence of coefficients $\{s_Q\}_{Q\in\mathcal{D}}$ satisfying

$$\Big[\sum_{\nu\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}_\nu}|s_Q|^p\Big)^{q/p}\Big]^{1/q} < \infty,$$

the series $\sum_{Q\in\mathcal{D}} s_Q a_Q$ converges in $\mathcal{S}'_\alpha$, then its sum f is in $\dot{B}^{\sigma,L_\alpha,m}_{p,q}$.
Moreover,

$$\|f\|_{\dot{B}^{\sigma,L_\alpha,m}_{p,q}} \lesssim \Big[\sum_{\nu\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}_\nu}|s_Q|^p\Big)^{q/p}\Big]^{1/q}.$$

Thank you!