# Spectral Invariance of Non-Smooth Pseudodifferential Operators

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# What are Pseudodifferential Operators

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- generalize differential operators, e.g. $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$,
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## Pseudodifferential Operators are a powerful tool

- in the index theory (Atiyah-Singer index theorem)
- for the study of differential operators,
- in the theory of partial differential equations, e.g. for
  - regularity results,
  - the existence of solutions.
- many other applications, e.g. denoising the noice recorded inside an MRI machine (V. Turunen)

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$$(1-\Delta)u = f \qquad \Rightarrow u = (1-\Delta)^{-1} f$$

# Smooth Pseudodifferential Operators

In the following let $n \in \mathbb{N}$ and $0 \leq \rho \leq 1$.

**Definition (smooth case)**

Let $m \in \mathbb{R}$. Then $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is in the *symbol-class* $S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if for all $\alpha, \beta \in \mathbb{N}_0^n$

$$\|\partial_\xi^\alpha \partial_x^\beta p(.,\xi)\|_\infty \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|} \qquad \forall \xi \in \mathbb{R}^n.$$

The function $p$ is called *(smooth) symbol* of the *order* $m$.

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## Definition

The linear operator $p(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, which is defined as

$$p(x, D_x)u(x) := \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n),\ x \in \mathbb{R}^n$$

is called *pseudodifferential operator*.

# Non-Smooth Pseudodifferential Operators

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Let $0 < s \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 \leq \rho \leq 1$. Then $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is in the *symbol-class* $C^{\tilde{m},s} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ if and only if for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$

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# Continuous Results of Pseudodifferential Operators

In the following let $1 < q < \infty$.

**Theorem (non-smooth case - Marschall (1985))**

*Let* $\tau > \frac{n}{2}$, $\tau \notin \mathbb{N}_0$, $m \in \mathbb{R}$ *and* $P \in OPC^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. *Then*

$$P : H_2^{m+s}(\mathbb{R}^n) \to H_2^s(\mathbb{R}^n) \qquad \text{is continuous for all } \frac{n}{2} - \tau < s < \tau.$$

Here $H_q^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : a(x, D_x)u \in L^q(\mathbb{R}^n)\}$, where $a(x,\xi) := (1 + |\xi|^2)^{s/2}$.

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**Theorem (non-smooth case - Marschall (1987))**

*Let $\tau > 0$, $\tau \notin \mathbb{N}_0$, $m \in \mathbb{R}$ and $P \in OPC^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

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$$Pu = f \qquad \Rightarrow \qquad u = P^{-1} f$$

# Characterisation of Smooth Pseudodifferential Operators:

**Theorem (smooth case - Ueberberg (1988))**

*Let $m \in \mathbb{R}$ und $P : H_q^m(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be linear and bounded such that*

$$\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta P : H_q^{r+m-\rho|\alpha|}(\mathbb{R}^n) \to H_q^r(\mathbb{R}^n)$$

*is continuous for all $r \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}_0^n$. Then $P \in OPS^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$.*

Here $\mathrm{ad}(-ix_j)P := -ix_jP + Pix_j$, $\mathrm{ad}(D_{x_j})P := D_{x_j}P - PD_{x_j}$ and $D_{x_j} := (-i)^{-1}\partial_{x_j}$.

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**Remark**

$$\mathrm{ad}(-ix)^\alpha\, \mathrm{ad}(D_x)^\beta \mathrm{OP}(p) = \mathrm{OP}(\partial^\alpha_\xi D^\beta_x p) \in \mathrm{OP}S^{m-\rho|\alpha|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n) \qquad \forall p \in S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

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**Theorem (Abels, P.(2017))**

*Let* $m \in \mathbb{R}$, $\rho \in \{0,1\}$, $\tilde{m} \in \mathbb{N}_0$, $\tilde{m} > \frac{n}{q}$, $N \in \mathbb{N}_0 \cup \{\infty\}$, $\tilde{N} := N - (n+1)$.
*Let* $P : H_q^m(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *be a linear and bounded operator such that*

$$\mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^\beta P : H_q^{m-\rho|\alpha|}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \qquad \textit{is continuous}$$

*for all* $\alpha, \beta \in \mathbb{N}_0^n$ *with* $|\alpha| \leq N$ *and* $|\beta| \leq \tilde{m}$. *Then we have for* $0 < \tau + k \leq \tilde{m} - \frac{n}{q}$ *with* $0 < \tau < 1, k \in \mathbb{N}_0$:

$$P \in OPC^{k,\tau} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N} - 1).$$

# Characterisation of Non-Smooth Pseudodifferential Operators

Sketch of the proof ($m = 0$, $\rho = 0$):
Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ on $B_{\frac{1}{2}}(0)$ and supp $\varphi \subseteq \overline{B_1(0)}$. Let

$$P_\varepsilon := \tilde{p}_\varepsilon(X, D_x) \qquad \text{and} \qquad Q_\varepsilon := q_\varepsilon(X, D_x) \qquad \text{for } 0 < \varepsilon \leq 1,$$

where $\tilde{p}_\varepsilon(x,\xi) := \varphi(\varepsilon x)$, $q_\varepsilon(x,\xi) := \varphi(\varepsilon\xi)\ \forall x, \xi \in \mathbb{R}^n$. Denote the linear operator $T_\varepsilon$ by

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The proof is divided into three different parts:

1. Writing $T_\varepsilon$ as a pseudodifferential operator with double symbol
2. Reducing the double symbol to a ordinary symbol $p_\varepsilon$ of $T_\varepsilon$.
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Differences to the smooth case:

Step 2: Prove, that $p_{\varepsilon,0}(x, \xi, y) := e^{-ix\cdot\xi} T_\varepsilon(e^{ix\cdot\xi} g_y)(x)$, where $g_y := g(x - y)$, $g \in \mathcal{S}(\mathbb{R}^n)$ with $g(0) = 1$ is a double symbol, i.e.

$$\|\partial_\xi^\alpha D_y^\gamma p_{\varepsilon,0}(., \xi, y)\|_{C^{k,\tau}} \leq C \|\partial_\xi^\alpha D_y^\gamma p_{\varepsilon,0}(., \xi, y)\|_{H_q^{\tilde{m}}} \leq C_{\alpha,\gamma}$$

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$\Rightarrow$ Loss of regularity with respect to $x$, since

- $\mathrm{ad}(-ix)^\alpha \mathrm{ad}(D_x)^\beta T \in \mathscr{L}(L^q)$ for all $|\alpha| \leq N$, $|\beta| \leq \tilde{m}$,
- Sobolev-Embeddings.

**Theorem (non-smooth case - Abels, P.(2016))**

*Let* $P \in OPC^{\tau}S^{0}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$, $\tau > 0$, $\tau \notin \mathbb{N}$ *with* $P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$. *Let* $\hat{m} := \max\{k \in \mathbb{N}_0 : \tau - k > \frac{n}{2}\}$, *then we have:*

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# Spectral Invariance of Non-Smooth Pseudodifferential Operators

## Theorem (non-smooth case - Abels, P.(2016))

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## Theorem (non-smooth case - Abels, P.(2016))

*Let* $1 < q_0 < \infty$, $0 < \tau < 1$, $\tilde{m}, \hat{m} \in \mathbb{N}_0$ *with* $\tilde{m} \geq \hat{m} > n/q_0$, $M \in \mathbb{N}_0$ *with* $M \leq \tilde{m} - \hat{m}$. *Define* $\tilde{M} := M - (n+1)$. *Let* $N \in \mathbb{N} \cup \{\infty\}$ *with* $N - M > \max\{n/2, n/q_0\}$. *Let* $P \in OPC^{\tilde{m},\tau} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ *with* $P^{-1} \in \mathscr{L}(L^{q_0}(\mathbb{R}^n))$. *Then*

$$P^{-1} \in OPC^r S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \qquad \forall\, 0 < r \leq \hat{m} - \frac{n}{q_0}, r \notin \mathbb{N}.$$

*If* $\tilde{M} - 1 > n/\tilde{q}$ *for a* $1 < \tilde{q} \leq 2$, *then*

$$P^{-1} \in \mathscr{L}(H^s_q(\mathbb{R}^n), H^s_q(\mathbb{R}^n)) \qquad \forall\, q \in [\tilde{q}; \infty) \cup \{q_0\}, -\hat{m} + \frac{n}{q_0} < s < \hat{m} - \frac{n}{q_0}.$$

# Thanks for your attention!

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