Hardy spaces associated with non-negative self-adjoint operators

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NPFSA-2017
Bedlewo - September, 2017

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Objectives

Develop Besov and Triebel-Lizorkin spaces, in particular, **Hardy spaces** and their frame characterization in the general setting of

- a metric measure space with the doubling property and
- in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property.

Apply these results to concrete settings.

The setting

(a) Let (M, ρ, μ) be a metric measure space such that (M, ρ) is locally compact and μ is a positive Radon measure obeying the doubling condition:

$$0 < \mu(B(x, 2r)) \le c\mu(B(x, r)) < \infty, \quad x \in M, r > 0,$$

which implies $\mu(B(x, \lambda r)) \le c\lambda^d \mu(B(x, r)), \; r > 0, \lambda > 1.$

(b) Let L be an essentially self-adjoint non-negative operator on $L^2(M, d\mu)$, mapping real-valued to real-valued functions, s.t. the (heat) kernel $p_t(x, y)$ of the associated semigroup

$$P_t = e^{-tL}$$

obeys

$$|p_t(x, y)| \le \frac{C \exp\{-\frac{c\rho^2(x,y)}{t}\}}{\left[\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))\right]^{1/2}} \quad \text{for } \; x, y \in M, \; t > 0.$$

Setting

(c) Hölder continuity: There exists a constant $\alpha > 0$ s.t

$$\left|p_t(x,y) - p_t(x,y')\right| \le C\Big(\frac{\rho(y,y')}{\sqrt{t}}\Big)^{\alpha} \frac{\exp\{-\frac{c\rho^2(x,y)}{t}\}}{\big[\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))\big]^{1/2}}$$

for $x, y, y' \in M$ and $t > 0$, whenever $\rho(y,y') \le \sqrt{t}$.

(d) Markov property:

$$\int_M p_t(x,y)d\mu(y) \equiv 1 \quad \text{for } t > 0.$$

Realization of the setting in Dirichlet spaces

A natural effective realization of the above setting appears in the general framework of strictly local regular Dirichlet spaces with a complete intrinsic metric, where it suffices to only verify

the **local Poincaré inequality** and
the **doubling condition on the measure**

and then the above general setting applies in full.

Realization of the setting in Dirichlet spaces

Operator $\Longrightarrow$ Quadratic form $\Longrightarrow$ Friedrich's extention

Let M is a locally compact separable metric space equipped with a positive Radon measure μ such that every open and nonempty set has positive measure.
Let L be a non-negative symmetric operator on (the real) $L^2(M,\mu)$ with domain $D(L)$, dense in $L^2(M,\mu)$.
Associate with L the symmetric non-negative form

$$\mathcal{E}(f,g) = \langle Lf, g\rangle = \mathcal{E}(g,f),\ \mathcal{E}(f,f) = \langle Lf, f\rangle \geq 0,$$

with domain $D(\mathcal{E}) = D(L)$. We consider on $D(\mathcal{E})$ the prehilbertian structure induced by

$$\|f\|_{\mathcal{E}}^2 = \|f\|_2^2 + \mathcal{E}(f,f),$$

which in general is not complete (not closed), but closable in L^2.

Realization of the setting (Cont.)

Friedrich's extention

Let $\overline{\mathcal{E}}$ and $D(\overline{\mathcal{E}})$ be the closure of $\mathcal{E}$ and its domain. $\overline{\mathcal{E}}$ gives rise to a self-adjoint extension $\bar{L}$ (the Friedrichs extension) of L with domain $D(\bar{L})$ consisting of all $f \in D(\overline{\mathcal{E}})$ for which there exists $u \in L^2$ such that $\overline{\mathcal{E}}(f,g) = \langle u, g\rangle$ for all $g \in D(\overline{\mathcal{E}})$ and $\bar{L}f = u$. Thus $\bar{L}$ is non-negative and self-adjoint, and

$$D(\overline{\mathcal{E}}) = D\big(\sqrt{\bar{L}}\big), \quad \overline{\mathcal{E}}(f,g) = \langle \sqrt{\bar{L}}f, \sqrt{\bar{L}}g\rangle.$$

From spectral theory, there is a self-adjoint strongly continuous contraction semigroup $P_t = e^{-t\bar{L}}$ on $L^2(M,\mu)$. Then

$$e^{-t\bar{L}} = \int_0^\infty e^{-\lambda t} dE_\lambda,$$

where E_λ is the spectral resolution associated with $\bar{L}$. Moreover, P_t has a holomorphic extension P_z, $\mathrm{Re}\, z > 0$.

Realization of the setting (Cont.)

Assumption: P_t is a Markov semigroup

If $0 \le f \le 1$ and $f \in L^2$, then $0 \le P_t f \le P_t 1 = 1$.
Then P_t can be extended as a contraction operator on L^p, $1 \le p \le \infty$, preserving positivity, satisfying $P_t 1 \le 1$, and hence yielding a strongly continuous contraction semigroup on L^p, $1 \le p < \infty$.

Sufficient condition (Beurling-Deny)

$\forall \varepsilon > 0 \; \exists \; \Phi_\varepsilon : \mathbb{R} \mapsto [-\varepsilon, 1+\varepsilon]$ s.t. Φ_ε is non-decreasing, $\Phi_\varepsilon(u') - \Phi_\varepsilon(u) \le u' - u$, if $u < u'$, $\Phi_\varepsilon(t) = t$ for $t \in [0,1]$, and

$$\Phi_\varepsilon(f) \in D(\overline{\mathcal{E}}) \;\text{ and }\; \overline{\mathcal{E}}(\Phi_\varepsilon(f), \Phi_\varepsilon(f)) \le \mathcal{E}(f,f), \quad \forall f \in D(L).$$

Under the above cond. $(D(\overline{\mathcal{E}}), \overline{\mathcal{E}})$ is called a **Dirichlet space.**

Realization of the setting (Cont.)

Assumption: $\overline{\mathcal{E}}$ is strictly local and regular

$\overline{\mathcal{E}}$ is **strictly local** $\iff \overline{\mathcal{E}}(f,g) = 0$ for $f, g \in D(\overline{\mathcal{E}})$ if $\operatorname{supp} f$ is compact and $g = const.$ on a neighborhood of $\operatorname{supp} f$.
$\overline{\mathcal{E}}$ is **regular** $\iff$ the space $\mathcal{C}_c(M)$ of continuous functions on M with compact support has the property that the algebra $\mathcal{C}_c(M) \cap D(\overline{\mathcal{E}})$ is dense in $\mathcal{C}_c(M)$ with respect to the sup norm, and dense in $D(\overline{\mathcal{E}})$ in the norm $\sqrt{\overline{\mathcal{E}}(f,f) + \|f\|_2^2}$.

Sufficient conditions for $\overline{\mathcal{E}}$ to be strictly local and regular

(i) $D(L)$ is a subalgebra of $\mathcal{C}_c(M)$ s.t. $\mathcal{E}(f,g) = \langle Lf, g\rangle = 0$ if $f, g \in D(L)$, $\operatorname{supp} f$ is compact, and g is constant on a neighbourhood of $\operatorname{supp} f$.
(ii) For any compact K and open set U s.t. $K \subset U$ there exists $u \in D(L)$, $u \geq 0$, $\operatorname{supp} u \subset U$, and $u \equiv 1$ on K.

Under the above assumptions, there exists a bilinear symmetric form $d\Gamma$ defined on $D(\overline{\mathcal{E}}) \times D(\overline{\mathcal{E}})$ with values in the signed Radon measures on M s.t.

$$\overline{\mathcal{E}}(f,g) = \int_M d\Gamma(f,g) \quad \text{and} \quad d\Gamma(f,f) \geq 0.$$

Moreover, if $D(L)$ is a subalgebra of $\mathcal{C}_c(M)$, then $d\Gamma$ is absolutely continuous with respect to μ, and

$$d\Gamma(f,g)(u) = \Gamma(f,g)(u)d\mu(u), \quad \Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf)$$

for $f, g \in D(L)$. Then $\overline{\mathcal{E}}(f,g) = \int_M \Gamma(f,g)(u)d\mu(u)$.

On $\mathbb{R}^n$ or Riemannian manifold $\Gamma(f,f) = |\nabla f|^2$.

Dirichlet spaces

Books:

- N. Bouleau, F. Hirsch, Dirichlet forms and analysis on Wiener space, de Gruyter Studies in Mathematics, 14, Walter de Gruyter and Co., Berlin, 1991.
- M. Fukushima, Y. Oshima, M. Takeda, Dirichlet forms and symmetric Markov processes. Second revised and extended edition, de Gruyter Studies in Mathematics, 19, Walter de Gruyter and Co., Berlin, 2011.

Realization of the setting (Cont.)

Definition of **intrinsic distance** on M:

$$\rho(x,y) = \sup\{h(x) - h(y) : h \in D(\overline{\mathcal{E}}) \cap \mathcal{C}_c(M), d\Gamma(h,h) \le d\mu\}.$$

where by definition

$$d\Gamma(h,h) \le d\mu \;\text{ if }\; d\Gamma(h,h) = \gamma(h) d\mu \;\text{ with }\; 0 \le \gamma(h) \le 1.$$

Assumption:

We assume that $\rho : M \times M \to [0,\infty]$ is a true metric that generates the original topology on M and (M,ρ) is a complete metric space.

Realization of the setting: Main conditions

1. (M, ρ, μ) obeys the **doubling condition**:

$$\mu(B(x, 2r)) \le c\mu(B(x, r), \quad x \in M, r > 0.$$

2. **Poincaré inequality**: $\exists C > 0$ s.t. $\forall B = B(x, r)$

$$\int_B |f - f_B|^2 \le Cr^2 \int_{2B} d\Gamma(f, f),$$

where $f_B = \frac{1}{B} \int_B f d\mu$.
These conditions are equivalent to: The semi-group P_t is a positive symmetric kernel operator with kernel $p_t(x, y)$ s.t.

$$\frac{c_1 \exp\{-\frac{c_2\rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \le p_t(x, y) \le \frac{c_3 \exp\{-\frac{c_4\rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}$$

Moreover, $p_t(x, y)$ is Hölder continuous.

Heat kernel associated with the Jacobi operator

Consider the interval $[-1,1]$ with measure $d\mu(x) = w(x)dx$, where

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1.$$

The Jacobi operator:

$$Lf(x) = -\frac{[w(x)a(x)f'(x)]'}{w(x)}, \quad a(x) = (1-x^2).$$

Eigenfunctions and eigenvalues: The (normalized) Jacobi polynomials P_k, $k \geq 0$, are eigenfunctions of the operator L, i.e.

$$LP_k = \lambda_k P_k \quad \text{with} \quad \lambda_k = k(k+\alpha+\beta+1).$$

Distance on $[-1,1]$: $\rho(x,y) = |\arccos x - \arccos y|$

Measure of balls:

$$|B(x,r)| = \mu(B(x,r)) \sim r(1-x+r^2)^{\alpha+1/2}(1+x+r^2)^{\beta+1/2}$$

Heat kernel associated with the Jacobi operator (Cont.)

Heat kernel (the kernel of the semi-group e^{-tL}, $t > 0$):

$$p_t(x,y) = \sum_{k\geq 0} e^{-\lambda_k t} P_k(x) P_k(y), \quad \lambda_k = k(k+\alpha+\beta+1)$$

Theorem. For any $0 < t \leq 1$ and $x, y \in [-1,1]$ we have

$$\frac{c_1 \exp\{-\frac{\rho^2(x,y)}{c_2 t}\}}{\left(|B(x,\sqrt{t})||B(y,\sqrt{t})|\right)^{1/2}} \leq p_t(x,y) \leq \frac{c_3 \exp\{-\frac{\rho^2(x,y)}{c_4 t}\}}{(|B(x,\sqrt{t})||B(y,\sqrt{t})|)^{1/2}}.$$

Heat kernel on the ball

Let $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| < 1\}$ with measure $d\nu(x) = w_\mu(x)dx$,

$$w_\mu(x) := (1 - \|x\|^2)^{\mu-1/2}, \quad \mu > -1/2, \ \ \|x\|^2 := x_1^2 + \cdots + x_d^2.$$

Differential operator: (here $\partial_i := \partial/\partial x_i$)

$$L := -\Delta + \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \partial_i \partial_j + (2\mu + d)\sum_{j=1}^{d} x_i \partial_i$$

The operator L is essentially self-adjoint and positive.

Eigenfunctions and eigenvalues: Let V_n^d the space for all polynomials of degree n in d variables which are orthogonal to lower degree polynomials in $L^2(\mathbb{B}^d, w_\mu)$. Then

$$LP = \lambda_n P \quad \text{for } P \in V_n^d \quad \text{with } \lambda_n = n(n + d + 2\mu - 1).$$

Metric on $\mathbb{B}^d$:

$$\rho(x, y) := \arccos\left\{\langle x, y\rangle + \sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}\right\}.$$

Heat kernel on the ball (Cont.)

Measure of balls: $\nu(B(x,r)) = |B(x,r)| \sim r^d(1-\|x\|^2+r^2)$

Heat kernel: Let $\{P_\alpha\}_{|\alpha|=n}$ be an orthonormal basis for V_n^d. Then the kernel $P_n(w_\mu; x, y)$ of the the orthogonal projector $\mathrm{Proj}_n : L^2(\mathbb{B}^d, w_\mu) \mapsto V_n^d$ can be written in the form

$$P_n(w_\mu; x, y) = \sum_{|\alpha|=n} P_\alpha(x) P_\alpha(y).$$

Heat kernel (the kernel of the semi-group e^{-tL}, $t > 0$)

$$p_t(x,y) = \sum_{n\ge 0} e^{-\lambda_n t} P_n(w_\mu; x, y).$$

Theorem. We have

$$\frac{c_1 \exp\left\{-\frac{\rho^2(x,y)}{c_2 t}\right\}}{(|B(x,\sqrt{t})||B(y,\sqrt{t})|)^{1/2}} \le p_t(x,y) \le \frac{c_3 \exp\left\{-\frac{\rho^2(x,y)}{c_4 t}\right\}}{(|B(x,\sqrt{t})||B(y,\sqrt{t})|)^{1/2}}.$$

Heat kernel on the simplex

Let $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \dots, x_d \geq 0, 1 - \|x\|_1 \geq 0\}$, where $\|x\|_1 := |x_1| + \dots + |x_d|$, with measure $d\nu(x) := w_\kappa(x)dx$,

$$w_\kappa(x) = x_1^{\kappa_1 - \frac{1}{2}} \cdots x_d^{\kappa_d - \frac{1}{2}} (1 - \|x\|_1)^{\kappa_{d+1} - \frac{1}{2}}, \quad \kappa_i > -1/2.$$

Differential operator: $(|\kappa| := \kappa_1 + \cdots + \kappa_d)$

$$L := -\sum_{i=1}^{d} x_i \partial_i^2 + \sum_{i=1}^{d} \sum_{j=1}^{d} x_i x_j \partial_i \partial_j - \sum_{i=1}^{d} \big(\kappa_i + \tfrac{1}{2} - (|\kappa| + \tfrac{d+1}{2}) x_i\big) \partial_i.$$

L is an essentially self-adjoint positive operator.
Eigenfunctions and eigenvalues: If V_n^d is the space of all polynomials of degree n which are orthogonal to lower degree polynomials in $L^2(\mathbb{T}^d, w_\kappa)$, then

$$LP = \lambda_n P \quad \text{for} \quad P \in V_n^d \quad \text{with} \quad \lambda_n := n(n+2l_\kappa), \quad l_\kappa := |\kappa| + \frac{d-1}{2}.$$

Heat kernel on the simplex (Cont.)

Metric on $\mathbb{T}^d$:

$$\rho(x,y) := \arccos\left\{\sqrt{x_1y_1} + \cdots + \sqrt{x_dy_d} + \sqrt{(1-\|x\|_1)(1-\|y\|_1)}\right\}$$

Measure of balls:

$$|B(x,r)| = \nu(B(x,r)) \sim r^d \prod_{i=1}^{d+1}(r^2+x_i)^{\kappa_i}$$

Heat kernel: Let $\{P_\alpha\}_{|\alpha|=n}$ be an orthonormal basis for V_n^d. The kernel of the orthogonal projector $\mathrm{Proj}_n : L^2(\mathbb{T}^d, w_\kappa) \mapsto V_n^d$ can be written in the form

$$P_n(w_\kappa; x, y) = \sum_{|\alpha|=n} P_\alpha(x)P_\alpha(y).$$

The heat kernel can be written as

$$p_t(x,y) = \sum_{n\geq 0} e^{-\lambda_n t} P_n(w_\kappa; x, y).$$

Heat kernel on the simplex (Cont.)

Theorem. For $0 < t \le 1$ and $x, y \in \mathbb{T}^d$

$$\frac{c_1 \exp\{-\frac{\rho^2(x,y)}{c_2 t}\}}{\big(|B(x,\sqrt{t})||B(y,\sqrt{t})|\big)^{1/2}} \le p_t(x,y) \le \frac{c_3 \exp\{-\frac{\rho^2(x,y)}{c_4 t}\}}{\big(|B(x,\sqrt{t})||B(y,\sqrt{t})|\big)^{1/2}}.$$

Other examples

- Classical setting on $\mathbb{R}^d$ with $L = -\Delta$
- Classical setting on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ with L being the Laplace-Beltrami operator
- Uniformly elliptic divergence form operators on $\mathbb{R}^d$
- Uniformly elliptic divergence form operators on subdomains of $\mathbb{R}^d$ with boundary conditions
- Riemannian manifolds and Lie groups. In particular, Compact Riemannian manifolds, Riemannian manifold with non-negative Ricci curvature, Compact Lie groups, Lie groups with polynomial growth and their homogeneous spaces, ...

Heat kernel theory: References

Heat kernel theory people: P. Auscher, M. Barlow, R. Bass, T. Coulhon, B. Davies, A. Grigoryan, E.M Ouhabaz, L. Saloff-Coste, A. Sikora, K. Sturm, N. Varopoulos, ...

Books:

- E. B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1989.
- A. Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics vol. 47, 2009.
- E.M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, 2005.
- L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, 2002.

Davies-Gaffney estimate:

$$|\langle P_t f_1, f_2\rangle| \le \exp\Big\{-\frac{c^\star r^2}{t}\Big\}\|f_1\|_2\|f_2\|_2, \quad t>0,$$

for all open sets $U_j \subset M$ and $f_j \in L^2(U_j)$, $j=1,2$, where $r := \rho(U_1, U_2)$ and $c^\star > 0$ is a constant.

Finite speed propagation property:

$$\big\langle \cos(t\sqrt{L})f_1, f_2\big\rangle = 0, \quad 0<\tilde{c}t<r, \quad \tilde{c} := \frac{1}{2\sqrt{c^\star}},$$

for all open sets $U_j \subset M$, $f_j \in L^2(U_j)$, $j=1,2$, where $r := \rho(U_1, U_2)$.

Key implications of the heat kernel properties (Cont.)

Let E_λ, $\lambda \geq 0$, be the spectral resolution associated with L. Then for any measurable and bounded function f on $\mathbb{R}_+$ the operator $f(L)$ is defined by

$$f(L) := \int_0^\infty f(\lambda) dE_\lambda$$

Let F_λ, $\lambda \geq 0$, be the spectral resolution associated with $\sqrt{L}$, i.e. $F_\lambda = E_{\lambda^2}$. Then

$$f(\sqrt{L}) := \int_0^\infty f(\lambda) dF_\lambda$$

Proposition. Let f be even, $\operatorname{supp} \hat{f} \subset [-A, A]$, $A > 0$, and $\hat{f} \in W_1^m$, $m > d$, i.e. $\|\hat{f}^{(m)}\|_{L^1} < \infty$. Then for $\delta > 0$ and $x, y \in M$

$$f(\delta\sqrt{L})(x, y) = 0 \quad \text{if} \quad \rho(x, y) > \tilde{c}\delta A.$$

Theorem. Suppose $f \in \mathcal{S}(\mathbb{R})$ and let f be even. Then $f(\delta\sqrt{L})$, $\delta > 0$, is an integral operator with kernel $f(\delta\sqrt{L})(x,y)$ satisfying

$$\left|f(\delta\sqrt{L})(x,y)\right| \le c_\sigma \frac{\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}}{\left(|B(x,\delta)||B(y,\delta)|\right)^{1/2}} \quad \forall \sigma > 0,$$

and if $\rho(y,y') \le \delta$

$$\left|f(\delta\sqrt{L})(x,y) - f(\delta\sqrt{L})(x,y')\right| \le c_\sigma \left(\frac{\rho(y,y')}{\delta}\right)^{\alpha} \frac{\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}}{\left(|B(x,\delta)||B(y,\delta)|\right)^{1/2}}$$

for some $\alpha > 0$.
Here $B(x,\delta)$ is the ball with center x and radius δ.

Sup-exponential localization

Theorem. For any $0 < \varepsilon < 1$ there exists $\kappa > 0$ and a compactly supported cut-off function $\varphi \in C^\infty$ s.t. for any $\delta > 0$

$$|\varphi(\delta\sqrt{L})(x,y)| \le \frac{c \exp\Big\{-\kappa\Big(\frac{\rho(x,y)}{\delta}\Big)^{1-\varepsilon}\Big\}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}}, \quad x, y \in M,$$

and if $\rho(y,y') \le \delta$

$$|\varphi(\delta\sqrt{L})(x,y) - \varphi(\delta\sqrt{L})(x,y')| \le \frac{c\Big(\frac{\rho(y,y')}{\delta}\Big)^{\alpha} \exp\Big\{-\kappa\Big(\frac{\rho(x,y)}{\delta}\Big)^{1-\varepsilon}\Big\}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}}.$$

Spectral spaces

Let F_λ, $\lambda \geq 0$, be the spectral resolution associated with $\sqrt{L}$, i.e. $F_\lambda = E_{\lambda^2}$ and hence $\sqrt{L} = \int_0^\infty \lambda dF_\lambda$.
The spectral space Σ_λ is defined by

$$\Sigma_\lambda := \{f \in L^2 : F_\lambda f = f\}.$$

This can be extended to define Σ_λ^p, $1 \leq p \leq \infty$:

$$\Sigma_\lambda^p := \{f \in L^p : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } [0,\lambda]\}.$$

Frames in the general setting: Properties

Decomposition system: $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$, $\mathcal{X} = \cup_{j\ge 0}\mathcal{X}_j$

Representation: for any $f \in L^p$, $1 \le p \le \infty$, with $L^\infty := \mathrm{UCB}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\psi}_\xi\rangle \psi_\xi = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \tilde{\psi}_\xi \quad \text{in} \;\; L^p.$$

Frame: The system $\{\tilde{\psi}_\xi\}$ is a frame for L^2:

$$c^{-1}\|f\|_2^2 \le \sum_{\xi\in\mathcal{X}} |\langle f, \tilde{\psi}_\xi\rangle|^2 \le c\|f\|_2^2, \quad \forall f \in L^2.$$

The same is true for $\{\psi_\xi\}$.

Frame Properties

Space localization: For any $\xi \in \mathcal{X}_j$, $j \geq 0$,

$$|\psi_\xi(x)| \leq c|B(\xi, b^{-j})|^{-1/2} \exp\{-\kappa(b^j\rho(x,\xi))^\beta\},$$

and if $\rho(x,y) \leq b^{-j}$

$$|\psi_\xi(x) - \psi_\xi(y)| \leq c|B(\xi, b^{-j})|^{-1/2}(b^j\rho(x,y))^\alpha \exp\{-\kappa(b^j\rho(x,\xi))^\beta\}.$$

Here $0 < \kappa < 1$ and $b > 1$ are constants. Same holds for $\tilde{\psi}_\xi$.

Spectral localization: $\psi_\xi, \tilde{\psi}_\xi \in \Sigma_b^p$ if $\xi \in \mathcal{X}_0$ and

$$\psi_\xi, \tilde{\psi}_\xi \in \Sigma^p_{[b^{j-2}, b^{j+2}]} \text{ if } \xi \in \mathcal{X}_j,\ j \geq 1,\ 0 < p \leq \infty.$$

Norms:

$$\|\psi_\xi\|_p \sim \|\tilde{\psi}_\xi\|_p \sim |B(\xi, b^{-j})|^{\frac{1}{p}-\frac{1}{2}} \quad \text{for } 0 < p \leq \infty.$$

Test functions: In our setting the class of test functions $\mathcal{S}$ is defined as the set of all functions $\phi \in \bigcap_{m\ge 0} D(L^m)$ s.t.

$$\mathcal{P}_m(\phi) := \sup_{x\in M}(1+\rho(x,x_0))^m \max_{0\le\nu\le m}|L^\nu\phi(x)| < \infty \quad \forall m \ge 0.$$

Here $x_0 \in M$ is selected arbitrarily and fixed once and for all.

Observe that if the function $\varphi \in \mathcal{S}(\mathbb{R})$ is real-valued and even, then $\varphi(\sqrt{L})(x,\cdot) \in \mathcal{S}$ and $\varphi(\sqrt{L})(\cdot,y) \in \mathcal{S}$.

The space of distributions $\mathcal{S}'$ on M is defined as the set of all continuous linear functionals on $\mathcal{S}$.

Definition of Besov spaces on $\mathbb{R}^d$

Let $\varphi_0, \varphi \in C^\infty(\mathbb{R}^d)$,

$$\operatorname{supp}\widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2\}, \quad |\widehat{\varphi}_0(\xi)| \ge c > 0 \quad \text{if} \quad |\xi| \le \frac{5}{3},$$

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\}, \quad |\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if} \quad \frac{3}{5} \le |\xi| \le \frac{5}{3}.$$

Then $|\widehat{\varphi}_0(\xi)| + \sum_{j\ge 1} |\widehat{\varphi}(2^{-j}\xi)| \ge c > 0, \quad \xi \in \mathbb{R}^d$.

Set $\varphi_j(x) := 2^{dj}\varphi(2^j x)$ for $j \ge 1$.

Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.

The inhomogeneous Besov space $B^s_{pq} = B^s_{pq}(\mathbb{R}^d)$ is defined as the set of all tempered distributions $f \in \mathcal{S}'$ s.t.

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j\ge 0} \Big(2^{sj} \|\varphi_j * f\|_{L^p} \Big)^q \Big)^{1/q} < \infty.$$

Definition of Besov spaces

Let $\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\varphi_0 \subset [0,2]$, $\varphi_0^{(\nu)}(0) = 0$ for $\nu \geq 1$, $\operatorname{supp}\varphi \subset [1/2, 2]$, and $|\varphi_0(\lambda)| + \sum_{j\geq 1} |\varphi(2^{-j}\lambda)| \geq c > 0$, $\lambda \in \mathbb{R}_+$.

Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ for $j \geq 1$.

Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$.

(i) The "classical" Besov space $B^s_{pq} = B^s_{pq}(L)$ is defined by

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j\geq 0}\Big(2^{sj}\|\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q}.$$

(ii) The "nonclassical" Besov space $\tilde{B}^s_{pq} = \tilde{B}^s_{pq}(L)$ is defined by

$$\|f\|_{\tilde{B}^s_{pq}} := \Big(\sum_{j\geq 0}\Big(\||B(\cdot, 2^{-j})|^{-s/d}\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q}.$$

Frame decomposition of Besov spaces

Theorem. Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Then for any $f \in B^s_{pq}$

$$\|f\|_{B^s_{pq}} \sim \Big(\sum_{j\ge 0} b^{jsq} \Big[\sum_{\xi\in\mathcal{X}_j} \|\langle f, \tilde\psi_\xi\rangle \psi_\xi\|_p^p \Big]^{q/p} \Big)^{1/q}$$

and for $f \in \tilde{B}^s_{pq}$

$$\|f\|_{\tilde{B}^s_{pq}} \sim \Big(\sum_{j\ge 0} \Big[\sum_{\xi\in\mathcal{X}_j} \Big(|B(\xi, b^{-j})|^{-s/d} \|\langle f, \tilde\psi_\xi\rangle \psi_\xi\|_p \Big)^p \Big]^{q/p} \Big)^{1/q}.$$

Here $b > 1$ is from the definition of the frames.

Definition of Triebel-Lizorkin spaces on $\mathbb{R}^d$

Let $\varphi_0, \varphi \in C^\infty(\mathbb{R}^d)$,

$$\operatorname{supp} \widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2\}, \quad |\widehat{\varphi}_0(\xi)| \ge c > 0 \quad \text{if} \quad |\xi| \le \frac{5}{3},$$

$$\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\}, \quad |\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if} \quad \frac{3}{5} \le |\xi| \le \frac{5}{3}.$$

Set $\varphi_j(x) := 2^{dj}\varphi(2^j x)$ for $j \ge 1$.

Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.

The inhomogeneous Triebel-Lizorkin space $F^s_{pq} = F^s_{pq}(\mathbb{R}^d)$ is defined as the set of all tempered distributions $f \in \mathcal{S}'$ s.t

$$\|f\|_{F^s_{pq}} := \Big\| \Big(\sum_{j \ge 0} \Big(2^{sj} |\varphi_j * f(\cdot)| \Big)^q \Big)^{1/q} \Big\|_{L^p} < \infty.$$

Triebel-Lizorkin space

Let $\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\varphi_0 \subset [0,2]$, $\varphi_0^{(\nu)}(0) = 0$ for $\nu \ge 1$, $\operatorname{supp}\varphi \subset [1/2, 2]$, and $|\varphi_0(\lambda)| + \sum_{j\ge 1} |\varphi(2^{-j}\lambda)| \ge c > 0$, $\lambda \in \mathbb{R}_+$.

Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ for $j \ge 1$.

Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$.

(i) The "classical" Triebel-Lizorkin space $F^s_{pq} = F^s_{pq}(L)$ is defined by

$$\|f\|_{F^s_{pq}} := \Big\| \Big(\sum_{j\ge 0} \Big(2^{js} |\varphi_j(\sqrt{L}) f(\cdot)| \Big)^q \Big)^{1/q} \Big\|_{L^p}.$$

(ii) The "nonclassical" Triebel-Lizorkin space $\tilde{F}^s_{pq} = \tilde{F}^s_{pq}(L)$ is defined by

$$\|f\|_{\tilde{F}^s_{pq}} := \Big\| \Big(\sum_{j\ge 0} \Big(|B(\cdot, 2^{-j})|^{-s/d} |\varphi_j(\sqrt{L}) f(\cdot)| \Big)^q \Big)^{1/q} \Big\|_{L^p}.$$

Theorem. Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Then for any $f \in F^s_{pq}$

$$\|f\|_{F^s_{pq}} \sim \Big\| \Big(\sum_{j \ge 0} b^{jsq} \sum_{\xi \in \mathcal{X}_j} \big[|\langle f, \tilde{\psi}_\xi \rangle| |\psi_\xi(\cdot)| \big]^q \Big)^{1/q} \Big\|_{L^p}.$$

and for $f \in \tilde{F}^s_{pq}$

$$\|f\|_{\tilde{F}^s_{pq}} \sim \Big\| \Big(\sum_{\xi \in \mathcal{X}} \big[|B(\xi, b^{-j})|^{-s/d} |\langle f, \tilde{\psi}_\xi \rangle| |\psi_\xi(\cdot)| \big]^q \Big)^{1/q} \Big\|_{L^p}.$$

Here $b > 1$ is from the definition of the frames.

Publications

1. T. Coulhon, G. Kerkyacharian, P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces, J. Fourier Anal. Appl. 18 (2012), no. 5, 995-1066.

2. G. Kerkyacharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, Trans. Amer. Math. Soc. 367 (2015), no. 1, 121–189.

Classical Hardy spaces

Maximal operators: Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'$ one defines

$$M_\varphi f(x) := \sup_{t>0} |\varphi_t * f(x)| \text{ with } \varphi_t(x) := t^{-n}\varphi(t^{-1}x), \quad \text{and}$$

$$M^*_{\varphi,a} f(x) := \sup_{t>0} \sup_{y\in\mathbb{R}^n, |x-y|\le at} |\varphi_t * f(y)|, \quad a \ge 1.$$

Recall the grand maximal operator. Write

$$\mathcal{P}_N(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N \max_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|$$

and denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S} : \mathcal{P}_N(\varphi) \le 1\}.$$

The grand maximal operator is defined by

$$\mathcal{M}_N f(x) := \sup_{\varphi\in\mathcal{F}_N} M^*_{\varphi,1} f(x), \quad f \in \mathcal{S}'.$$

Definition.
The space $H^p = H^p(\mathbb{R}^n)$, $0 < p \le 1$, is defined as the set of all bounded distributions $f \in \mathcal{S}'$ s.t. the Poisson maximal function $\sup_{t>0} |P_t * f(x)|$ belongs to L^p; the quasi-norm on H^p is defined by

$$\|f\|_{H^p} := \Big\| \sup_{t>0} |P_t * f(\cdot)| \Big\|_{L^p}.$$

Proposition. Let $0 < p \le 1$, $a \ge 1$, and assume $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \varphi \ne 0$. Then for any $N \ge \lfloor \frac{n}{p} \rfloor + 1$

$$\|f\|_{H^p} \sim \|M^*_{\varphi,a} f\|_{L^p} \sim \|\mathcal{M}_N f\|_{L^p}, \quad \forall f \in H^p.$$

Atomic Hardy spaces on $\mathbb{R}^n$

Atoms. A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball B s.t.

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_{L^\infty} \le |B|^{-1/p}$, and

(iii) $\int_{\mathbb{R}^n} x^\alpha a(x)dx = 0$ for all α with $|\alpha| \le n(p^{-1} - 1)$.

The atomic Hardy space H^p_A, $0 < p \le 1$, is defined as the set of all $f \in \mathcal{S}'$ that can be represented in the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty,$$

$\{a_j\}$ are atoms, and the convergence is in $\mathcal{S}'$. Set

$$\|f\|_{H^p_A} := \inf_{f=\sum_j \lambda_j a_j} \Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p}, \quad f \in H^p_A.$$

Theorem (R. Coifman, R. Latter). One has $H^p = H^p_A$, $0 < p \le 1$, and

$$\|f\|_{H^p_A} \sim \|f\|_{H^p} \quad \text{for} \quad f \in H^p.$$

References:

1. R. Coifman, A real variable characterization of H^p, Studia Math. 51 (1974), 269–274.

2. R. Latter, A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms, Studia Math. 62 (1978), 93–101.

3. C. Fefferman, E. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–193.

4. E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

Hardy spaces
in the general setting
Version I

Maximal operators and definition of Hardy spaces

Definition. A function $\varphi \in \mathcal{S}(\mathbb{R})$ is called admissible if φ is real-valued and even. Norms on admissible functions in $\mathcal{S}(\mathbb{R})$:

$$\mathcal{N}_N(\varphi) := \sup_{u\geq 0}(1+u)^N \max_{0\leq m\leq N} |\varphi^{(m)}(u)|, \quad N \geq 0.$$

Maximal operators: Let $\varphi \in \mathcal{S}(\mathbb{R})$ be admissible. For any $f \in \mathcal{S}'$ we define

$$M(f;\varphi)(x) := \sup_{t>0} |\varphi(t\sqrt{L})f(x)|,$$

$$M_a^*(f;\varphi)(x) := \sup_{t>0} \sup_{y\in M, \rho(x,y)\leq at} |\varphi(t\sqrt{L})f(y)|, \quad a \geq 1,$$

$$M_\gamma^{**}(f;\varphi)(x) := \sup_{t>0} \sup_{y\in M} \frac{|\varphi(t\sqrt{L})f(y)|}{\left(1+\frac{\rho(x,y)}{t}\right)^\gamma}, \quad \gamma > 0.$$

The grand maximal operator: Denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}) : \varphi \text{ is admissible and } \mathcal{N}_N(\varphi) \le 1\}.$$

The grand maximal function is defined by

$$\mathcal{M}_N(f)(x) := \sup_{\varphi \in \mathcal{F}_N} \sup_{t>0} \sup_{y \in M, \rho(x,y) \le t} |\varphi(t\sqrt{L})f(y)|,$$

where $N > 0$ is sufficiently large (to be specified).

The Hardy spaces H^p, $0 < p \le 1$

Definition. The Hardy space H^p, $0 < p \le 1$, is defined as the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{H^p} := \Big\| \sup_{t>0} |e^{-t^2 L} f(\cdot)| \Big\|_{L^p} < \infty.$$

Theorem. Let $0 < p \le 1$. Then for any $N > 6d/p + 3d/2 + 2$, $\gamma > 2d/p$, $a \ge 1$, and an admissible $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi(0) \ne 0$ we have for all $f \in \mathcal{S}'$

$$\|f\|_{H^p} \sim \|\mathcal{M}_N(f)\|_{L^p} \sim \|M(f;\varphi)\|_{L^p} \sim \|M_a^*(f;\varphi)\|_{L^p} \sim \|M_\gamma^{**}(f;\varphi)\|_{L^p}.$$

Also,

$$\Big\| \sup_{t>0} \Big| e^{-t\sqrt{L}} f(\cdot) \Big| \Big\|_{L^p} \sim \|f\|_{H^p}.$$

Atomic Hardy spaces H_A^p, $0 < p \le 1$, $\mu(M) = \infty$

Definition. Let $0 < p \le 1$ and $n := \lfloor d/2p \rfloor + 1$. A function $a(x)$ is called an atom associated with the operator L if there exists a function $b \in D(L^n)$ and a ball B of radius $r = r_B > 0$ such that

(i) $a = L^n b$,

(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, 1, \dots, n$, and

(iii) $\|L^k b\|_\infty \le r^{2(n-k)} |B|^{-1/p}$, $k = 0, 1, \dots, n$.

Definition. The atomic Hardy space H_A^p is defined as the set of all distributions $f \in \mathcal{S}'$ that can be represented in the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty,$$

$\{a_j\}$ are atoms, and the convergence is in $\mathcal{S}'$. We set

$$\|f\|_{H_A^p} := \inf_{f = \sum_{j\ge 1} \lambda_j a_j} \Big(\sum_{j \ge 1} |\lambda_j|^p \Big)^{1/p}, \quad f \in H_A^p.$$

(i) Atomic decomposition of Hardy spaces:

Theorem 1. We have $H^p = H^p_A$, $0 < p \le 1$, and

$$\|f\|_{H^p_A} \sim \|f\|_{H^p} \quad \text{for} \quad f \in H^p.$$

(ii) Littlewood-Paley characterization of Hardy spaces

Theorem 2. We have $H^p = F^0_{p2}$, $0 < p \le 1$, and

$$\|f\|_{H^p} \sim \|f\|_{F^0_{p2}} \quad \text{for} \quad f \in H^p.$$

Hardy spaces
in the general setting
Version II

The setting

(a) Let (M, ρ, μ) be a metric measure space such that (M, ρ) is locally compact and μ is a positive Radon measure obeying the doubling condition:

$$0 < \mu(B(x, 2r)) \le c\mu(B(x, r)) < \infty, \quad x \in M, r > 0.$$

(b) Let L be a non-negative self-adjoint operator on $L^2(M, d\mu)$, mapping real-valued to real-valued functions s.t. the semigroup $P_t = e^{-tL}$, $t > 0$, associated with L consists of integral operators with (heat) kernel $p_t(x, y)$ statisfying

$$|p_t(x, y)| \le \frac{C \exp\{-\frac{c\rho^2(x,y)}{t}\}}{\left[\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))\right]^{1/2}}, \quad \forall x, y \in \tilde{M}, \ \forall t > 0,$$

where $\tilde{M} \subset M$, $\tilde{M}$ is independent of t, with $\mu(M \setminus \tilde{M}) = 0$.

The Hardy spaces H^p, $0 < p \le 1$

Definition. The maximal Hardy space H^p, $0 < p \le 1$, in the general setting described above is defined as the completion of the set of all function $f \in L^2(X)$ such that

$$\|f\|_{H^p} := \Big\| \sup_{t>0} |e^{-t^2 L} f(\cdot)| \Big\|_{L^p} < \infty.$$

Definition. The atomic Hardy space H^p_A, $0 < p \le 1$, is defined as follows. We say that $f = \sum_{j\ge1} \lambda_j a_j$ is an atomic representation of f if $\{\lambda_j\}_{j\ge1} \in \ell^p$, all a_j, $j = 1, 2, \dots,$ are atoms, and the series converges in L^2. We denote by $\mathbb{H}^p_A$ the space of all $f \in L^2(M)$ that have atomic representations with norm

$$\|f\|_{\mathbb{H}^p_A} := \inf_{f=\sum_{j\ge1} \lambda_j a_j} \Big(\sum_{j\ge1} |\lambda_j|^p \Big)^{1/p}, \quad f \in \mathbb{H}^p_A.$$

Now, H^p_A, $0 < p \le 1$, is defined as the completion of $\mathbb{H}^p_A$ with respect to the above norm.

Atomic decomposition of Hardy spaces

Theorem. We have $H^p = H^p_A$, $0 < p \le 1$, and

$$\|f\|_{H^p_A} \sim \|f\|_{H^p} \quad \text{for} \quad f \in H^p.$$

Idea of proof

In proving that $H^p = H^p_A$, $0 < p \le 1$, the hard part is to show that if $f \in H^p$, $0 < p \le 1$, then $f \in H^p_A$ and $\|f\|_{H^p_A} \le c\|f\|_{H^p}$.

Decomposition: If the function $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even, $\varphi(0) = 1$, and with Fourier transform $\hat{\varphi}$ obeying $\operatorname{supp}\hat{\varphi} \subset [-1, 1]$, then there exist even real-valued functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ with the properties: $\psi_0(0) = 1$, $\psi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots, N$,

$$\operatorname{supp}\hat{\psi}_0 \subset [-2N, 2N], \ \operatorname{supp}(\lambda^{-\nu}\psi(\lambda))^\wedge \subset [-2N, 2N], \ \nu = 0, \dots, N,$$

such that for any $f \in \mathcal{S}'$ and $j \in \mathbb{Z}$

$$f = \psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f + \sum_{k=j}^{\infty} \psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L}) - \varphi(2^{-k+1}\sqrt{L})]f,$$

where the convergence is in $\mathcal{S}'$.

Idea of proof (Cont.)

Compact supports: Write $\tilde{\varphi} := \varphi_0$ and $\tilde{\psi}(\lambda) := \varphi(\lambda) - \varphi(2\lambda)$.
Denoted

$$\varphi_k := \varphi(2^{-k}\sqrt{L}), \ \tilde{\varphi}_k := \psi_0(2^{-k}\sqrt{L}),$$
$$\psi_k := \psi(2^{-k}\sqrt{L}), \ \tilde{\psi}_k := \tilde{\psi}(2^{-k}\sqrt{L}).$$

Claim: There exists a constant $\tau > 1$ s.t.

$$\operatorname{supp} \varphi_k(x,\cdot), \ \operatorname{supp} \tilde{\varphi}_k(x,\cdot), \ \operatorname{supp} \psi_k(x,\cdot), \ \operatorname{supp} \tilde{\psi}_k(x,\cdot) \subset B(x, \tau 2^{-k}),$$

and

$$\operatorname{supp}[L^{-\nu}\psi(2^{-k}\sqrt{L})](x,\cdot) \subset B(x, \tau 2^{-k}), \quad \nu = 0, 1, \dots, N.$$

Idea of proof (Cont.)

Whitney decomposition: Assume that Ω is an open subset of M, $\Omega \neq M$, and denote $\rho(x) := \mathrm{dist}(x, \Omega^c)$. Then there exist a constant $K > 0$ ($K = 70^d c_0^2$ will do) and a sequence of points $\{\xi_j\}_{j\in\mathbb{N}}$ in Ω with the following properties, where $\rho_j := \mathrm{dist}(\xi_j, \Omega^c)$:

(a) $\Omega = \cup_{j\in\mathbb{N}} B(\xi_j, \rho_j/2)$.

(b) $\{B(\xi_j, \rho_j/5)\}$ are disjoint.

(c) If $B\left(\xi_j, \frac{3\rho_j}{4}\right) \cap B\left(\xi_\nu, \frac{3\rho_\nu}{4}\right) \neq \emptyset$, then $7^{-1}\rho_\nu \le \rho_j \le 7\rho_\nu$.

(d) For every $j \in \mathbb{N}$ there are at most K balls $B\left(\xi_\nu, \frac{3\rho_\nu}{4}\right)$ intersecting $B\left(\xi_j, \frac{3\rho_j}{4}\right)$.

Publications

1. S. Dekel, G. Kerkyacharian, Kyriazis, P. Petrushev, Hardy spaces associated with non-negative self-adjoint operators, Studia Math. 239 (2017), 17–54.

2. S. Dekel, G, Kyriazis, G. Kerkyacharian, P. Petrushev, Compactly supported frames for spaces of distributions associated with non-negative self-adjoint operators, Studia Math. 225 (2014), 115–163.

Related Publications

1. X. Duong, Ji Li, Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus, J. Funct. Anal. 264 (2013), 1409–1437.

2. J. Dziubanski, M. Preisner, Hardy spaces for semigroups with Gaussian bounds, arXiv:1606.01064

3. S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, Mem. Amer. Math. Soc. 214 (2011), no. 1007.

4. R. Jiang, D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates, Commun. Contemp. Math. 13 (2011), 331–373.

5. L. Song, L. Yan, A maximal function characterization for Hardy spaces associated to nonnegative self-adjoint operators satisfying Gaussian estimates, Adv. Math. 287 (2016), 463–484.

Conclusion

There are two points to be made:

- Although not the most general our setting allows for a lot of flexibility. In particular, it allows us to deal with different geometries, compact and noncompact setups, and with nontrivial weights.
- Furthrmore, our setting allows to develop the theory of Hardy spaces, Besov and Triebel-Lizorkin spaces with complete range of indices, in almost complete generality as in the classical case on $\mathbb{R}^n$, and to cover a great deal of classical and nonclassical settings.

END