Convergence of Spline Projections

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Motivation - Conditional Expectation as spline projection

Let $\mathcal{A}_n$ be an increasing sequence of σ-algebras on $[0, 1]$, $\mathcal{A}_n$ be generated by a finite partition $(A_j^{(n)})_j$ of $[0, 1]$ into intervals and λ the Lebesgue measure.

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Then, the conditional expectation $\mathbb{E}_n := \mathbb{E}(\cdot|\mathcal{A}_n)$ is given by local averaging:

$$\mathbb{E}_n g = \sum_j \lambda(A_j^{(n)})^{-1} \Big(\int g \cdot \mathbb{1}_{A_j^{(n)}} \,\mathrm{d}\lambda \Big) \mathbb{1}_{A_j^{(n)}}.$$

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It is the orthogonal projection (w.r.t $L^2(\lambda)$) onto the space of piecewise constant functions w.r.t to $\mathcal{A}_n$.

Properties of $\mathbb{E}_n$

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3. Unconditionality of Martingale Differences in L^p, $p \in (1, \infty)$:

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Those estimates do *not* depend on the shape of $\mathcal{A}_n$.

Spline Projections - Definition

Similarly to projections $\mathbb{E}_n$ onto piecewise constant functions, we can introduce (for any positive integer d)

$$S_n = \text{the space of functions that are polynomials of degree } d \text{ on } A_j^{(n)} \text{ for any } j \text{ and are } d-1 \text{ times continuously differentiable at the boundary points of } A_j^{(n)},$$

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The generalization of the above properties for martingales to splines was studied extensively by various authors including S. Bočkarev, Z. Ciesielski, C. de Boor, G. Gevorkyan, A. Kamont, restricting either the degree d and/or the shape of $\mathcal{A}_n$.

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The operator P_n

Similar to the expansion

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we can write $P_n g$ using the B-spline basis $(N_j^{(n)})$ of S_n:

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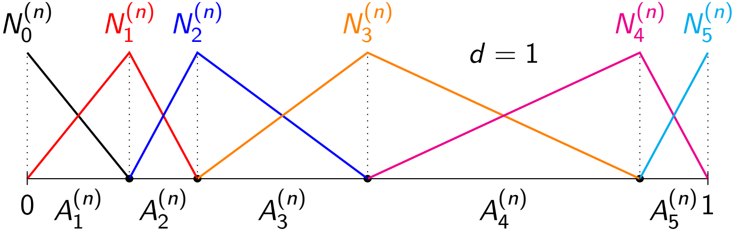
In the piecewise linear case ($d = 1$), B-splines look like:



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- for general d, the localization of $(N_j^{(n)})$ is not as sharp as the localization of $\mathbb{1}_{A_j^{(n)}}$, so $(a_{ij}^{(n)})$ is not diagonal,
- estimates for P_n are intimately tied to sharp decay estimates for $a_{ij}^{(n)}$ away from the diagonal.

Sharp estimates for $a_{ij}^{(n)}$

- Shadrin's theorem (BDL1) is equivalent to the uniform estimate

$$|a_{ij}^{(n)}| \leq C_d \frac{q^{|i-j|}}{\max\big(\lambda(\operatorname{supp} N_i^{(n)}), \lambda(\operatorname{supp} N_j^{(n)})\big)}, \qquad q \in (0,1).$$

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- Properties (AE) and (UCLp) can be deduced from the refinement

$$|a_{ij}^{(n)}| \leq C_d \frac{q^{|i-j|}}{\lambda\big(\operatorname{conv}(\operatorname{supp} N_i^{(n)} \cup \operatorname{supp} N_j^{(n)})\big)}, \qquad q \in (0,1).$$

Periodic setting

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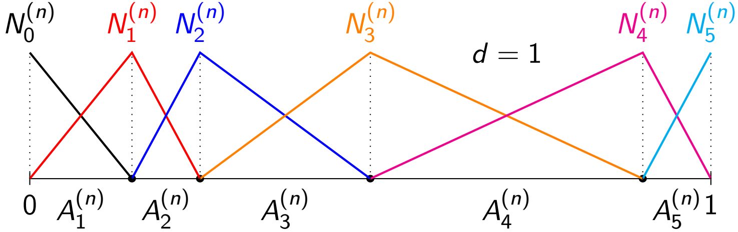
Similarly as in the non-periodic setting, one can define periodic B-spline functions $(\hat{N}_j^{(n)})$ and write

$$\hat{P}_n g = \sum_{i,j} \hat{a}_{ij}^{(n)} \Big(\int g \cdot \hat{N}_j^{(n)} \,\mathrm{d}\lambda \Big) \hat{N}_i^{(n)}.$$

Pictures: compare non-periodic to periodic B-splines

Non-periodic for $d = 1$:



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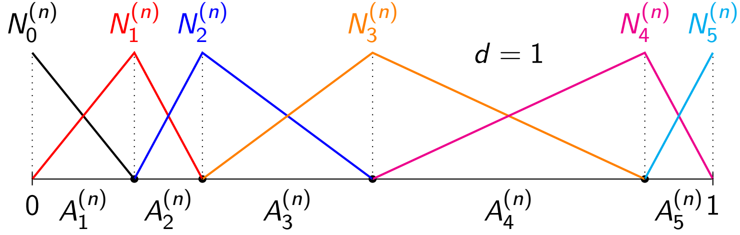

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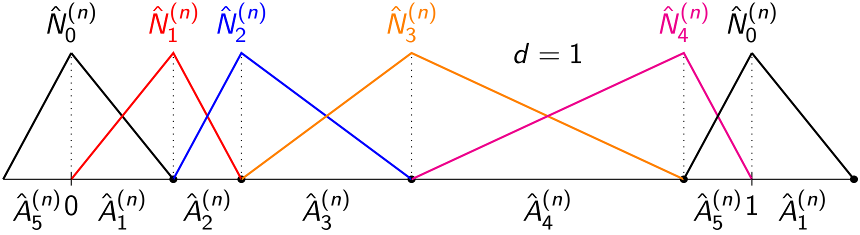

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   and the fact that most of the B-splines $(\hat{N}_i^{(n)})$ and $(N_i^{(n)})$ coincide (see picture).