Spectra of weighted composition operators on spaces of analytic functions

by

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Coauthors and references

This is a joint work with

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T. Eklund, M. Lindström, P. Mleczko
Spectral properties of weighted composition operators on the Bloch and Dirichlet spaces
Studia Math. 232 (2016), 95–112.

T. Eklund, M. Lindström, P. Mleczko, M. Rzeczkowski
Spectra of weighted composition operators on abstract Hardy spaces
submitted (2017), 13 pp.

Introduction

Composition operators

Let $\varphi \in H(\mathbb{D})$ be a self-map of $\mathbb{D}$, i.e., $\varphi(\mathbb{D}) \subset \mathbb{D}$. The map

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$$

given by

$$C_\varphi f(z) = f \circ \varphi(z), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}$$

is called a composition operator.

Composition operators – properties

- Boundedness (due to Littlewood, '20 of XX century) – every self-map of $\mathbb{D}$ generates a bounded composition operator $C_\varphi : H^p \to H^p$ (irrespective of p, $p \in [1, \infty]$).
- Compactness (due to J. Shapiro, '80 of XX century) – a composition operator C_φ is compact on H^p (irrespective of p, $p \in [1, \infty)$) if and only if

$$\lim_{|z| \to 1^-} \frac{N_\varphi(z)}{\log \frac{1}{|z|}} = 0,$$

where N_φ is a Nevanlinna counting function

Weighted composition operators

Let $\varphi \in H(\mathbb{D})$ be a self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the map

$$uC_\varphi \colon H(\mathbb{D}) \to H(\mathbb{D})$$

is called a weighted composition operator.

A map uC_φ is a bounded operator on H^p is and only if u is a bounded function.

Research task

Challenge

Study the spectral properties of weighted composition operators on spaces of holomorphic functions (*e.g., Hardy, Bloch spaces etc*).

Results

Notation

- For a self-map of $\mathbb{D}$ the symbol φ_n denotes the n-th iterate of φ, i.e.,

$$\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

- For any n

$$(uC_\varphi)^n f(z) = u(z)u(\varphi(z)) \dots u(\varphi_{n-1}(z)) f(\varphi_n(z)) \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}$$

and

$$(uC_\varphi)^n = u_{(n)} C_{\varphi_n},$$

where

$$u_{(n)} = \prod_{j=0}^{n-1} u \circ \varphi_j \in H(\mathbb{D}), \quad n \in \mathbb{N}.$$

Köthe function spaces

X is a Köthe function spaces if

- $X \subset L^0(\Omega, \Sigma, \mu)$ – the space of real valued measurable functions on Ω. The order $|x| \leq |y|$ means $|x(\omega)| \leq |y(\omega)|$ for μ-almost all $\omega \in \Omega$.
- There exists $u \in X$ with $u > 0$ μ-a.e. on Ω and $|x| \leq |y|$ with $x \in L^0(\Omega)$ and $y \in X$ implies $x \in X$ with $\|x\|_X \leq \|y\|_X$.
- If $x \in X$, then for any g equimeasurable with f, $\|f\|_X = \|g\|_X$

We will consider complex Köthe function spaces.

The role model for X are Lebesgue spaces, other important examples

- Orlicz spaces
- Lorentz spaces
- Marcinkiewicz spaces

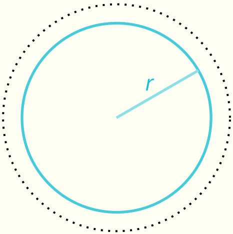
Köthe function space X is maximal (or has the Fatou property) if whenever $\{x_n\}$ is a norm bounded sequence in X such that $0 \leq x_n \uparrow x \in L^0(\Omega)$, then $x \in X$ and $\|x\| = \lim_{n\to\infty} \|x_n\|$.

Abstract Hardy spaces

Let X be the Köthe space on $\mathbb{T} := [0, 2\pi)$. We define abstract Hardy spaces $HX(\mathbb{D})$ in the following way.

$$HX(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{HX(\mathbb{D})} := \sup_{r \in [0,1)} \|f_r\|_X < \infty \right\}$$

where $f_r(t) := f(re^{it})$, $t \in \mathbb{T}$.



Multipliers of abstract Hardy spaces

For any function $u \in H(\mathbb{D})$, an operator

$$M_u\colon H(\mathbb{D}) \to H(\mathbb{D}), \quad M_u f(z) = u(z)f(z), \quad f \in H(\mathbb{D})$$

is called a multiplication operator.

Theorem

Let X be a maximal r.i. space on $\mathbb{T}$ and suppose that $u \in H(\mathbb{D})$. Then the following statements are equivalent:

- $uHX \subset HX$
- *$u \in HX$ and the operator M_u is bounded*
- $u \in H^\infty$.

Moreover, $\|M_u\colon HX \to HX\| = \|u\|_\infty$.

Automorphisms of the discs

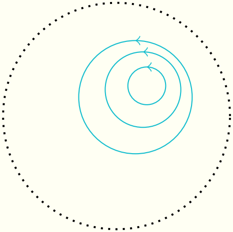
φ – nontrivial (i.e., not identity) automorhism of $\mathbb{D}$

- elliptic – φ has a unique fixed point in $\mathbb{D}$,
- parabolic – φ has a unique fixed (Denjoy–Wolf) point $a \in \partial\mathbb{D}$, $\varphi'(a) = 1$,
- hiperbolic – φ has two distinct fixed points in $\partial\mathbb{D}$, an attraxtive fixed (Denjoy–Wolf) point a and a repulsive fixed point b, $\varphi'(a) \in (0, 1)$ and $\varphi'(b) = 1/\varphi'(a)$.

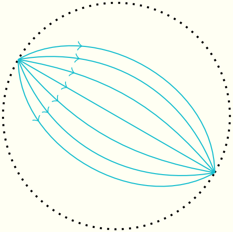
When φ is a parabolic or hyperbolic automorphism, then

$$\lim_{n\to\infty} \left(1 - |\varphi_n(0)|\right)^{\frac{1}{n}} = \varphi'(a).$$

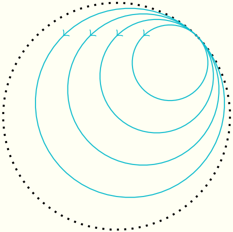
Automorphisms of the disc – pictures



elliptic



hyperbolic



parabolic

Abstract Hardy spaces – elliptic case

Theorem

Suppose that $u \in A(\mathbb{D})$ and φ is an elliptic automorphism with the unique fixed point $a \in \mathbb{D}$.

- *If there is a positive integer j such that $\varphi_j(z) = z$ for all $z \in \mathbb{D}$, then letting m be the smallest such integer, we have*

$$\sigma_{HX}(uC_\varphi) = \overline{\{\lambda \in \mathbb{C} \colon u_{(m)}(z) = \lambda^m \text{ for some } z \in \mathbb{D}\}}.$$

- *If $\varphi_n \neq \mathrm{Id}$ for every $n \in \mathbb{N}$ and if $uC_\varphi \colon HX \to HX$ is invertible, then*

$$\sigma_{HX}(uC_\varphi) = \{\lambda \in \mathbb{C} \colon |\lambda| = |u(a)|\}.$$

Abstract Hardy spaces – parabolic case

Theorem

Let X be a maximal r.i. space on $\mathbb{T}$ and $uC_\varphi : HX \to HX$ be an invertible composition operator and assume that φ is a parabolic automorphism of the disc $\mathbb{D}$ with a unique fixed point $a \in \partial\mathbb{D}$ and $u \in A(\mathbb{D})$. Then

$$r_{HX}(uC_\varphi) = |u(a)|$$

and

$$\sigma_{HX}(uC_\varphi) = \{z \in \mathbb{C} : |z| = |u(a)|\}.$$

Invertible weighted composition operators

Theorem

Let X be an r.i. space on $\mathbb{T}$ and suppose that $uC_\varphi : HX \to HX$ is a weighted composition operator. Then uC_φ is invertible if and only if $u \in H^\infty$, u is bounded away from zero on $\mathbb{D}$, and φ is an automorphism of $\mathbb{D}$. Moreover, if this is the case, then the inverse is also a weighted composition operator and

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}}.$$

P. Bourdon
Invertible weighted composition operators
Proc. Amer. Math. Soc. 142 (2013), 289–299.

Hardy–Lorentz spaces – stability of the spectrum

Given a non-atomic measure space (Ω, Σ, μ) and $p \in (1, \infty)$, $q \in [1, \infty]$, the Lorentz space $L^{p,q}$ on (Ω, Σ, μ) consists of all $L^0(\Omega)$ such that

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^{\mu(\Omega)} \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t} \right)^{1/q}, & q < \infty \\ \sup_{t \in (0, \mu(\Omega))} t^{1/p} f^{**}(t), & q = \infty, \end{cases}$$

where $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$, $t > 0$ and f^* denotes the non-increasing rearrangement of f (i.e., $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$, $t \geq 0$).

Recall that for $p \in (1, \infty)$, $q \in [1, \infty]$, the Hardy–Lorentz space $H^{p,q}$ is given by $H^{p,q} = HL^{p,q}$.

Theorem

Let φ be a holomorphic self-map of $\mathbb{D}$, $u \in H^\infty$, and $p, q \in (1, \infty)$. Then

$$\sigma_{H^{p,q}}(uC_\varphi) = \sigma_{H^p}(uC_\varphi).$$

Hardy–Lorentz spaces – automorphisms

Corollary

Let uC_φ be an invertible operator on Hardy–Lorentz space $H^{p,q}$, $p, q \in (1, \infty)$ and $u \in A(\mathbb{D})$. Then

- *if φ is a parabolic automorphism with fixed point $s \in \partial\mathbb{D}$, then*

$$\sigma_{H^{p,q}}(uC_\varphi) = \left\{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\right\},$$

- *if φ is a hyperbolic automorphism with attractive point $a \in \partial\mathbb{D}$ and repulsive point $b \in \partial\mathbb{D}$, then*

$$\sigma_{H^{p,q}}(uC_\varphi) = \left\{\lambda \in \mathbb{C} : \min\left\{\frac{|u(a)|}{\varphi'(a)^{1/p}}, \frac{|u(b)|}{\varphi'(b)^{1/p}}\right\} \leq |\lambda| \leq \max\left\{\frac{|u(a)|}{\varphi'(a)^{1/p}}, \frac{|u(b)|}{\varphi'(b)^{1/p}}\right\}\right\}.$$

O. Hyvärinen, I. Nieminen
Essential spectra of weighted composition operators with hyperbolic symbols
Concr. Oper. 2 (2015), 110–119.

Hardy–Lorentz spaces – non-automorphisms

Corollary

Assume that $u, \varphi \in H(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, and a is a Denjoy–Wolff point with $\varphi_n \to a$ uniformly in $\mathbb{D}$ as $n \to \infty$ and $\varphi'(a) < 1$. If u is bounded in $\mathbb{D}$ and continuous at a, then

$$\sigma_{H^{p,q}}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{u(a)|}{|\varphi'(a)|^{1/p}} \right\}.$$

O. Hyvärinen, I. Nieminen
Essential spectra of weighted composition operators with hyperbolic symbols
Concr. Oper. 2 (2015), 110–119.

C. Cowen, E. Ko, D. Thompson, F. Tian
Spectra of some weighted composition operators on H^2
Acta Sci. Math. (Szeged) 82 (2016), no. 1–2, 221–234.

Proofs ingredients – interpolation and interpolation

- Complex functions theory. A sequence $\{z_j\} \subset \mathbb{D}$ is called interpolating for $H^\infty(\mathbb{D})$ if for any bounded sequence of complex numbers $\{c_j\}$, there is a function $f \in H^\infty(\mathbb{D})$ such that $f(z_j) = c_j$. It is known that if $z_0 \in \mathbb{D}$, then the sequence of iterates $\{\varphi_n(z_0)\}$, $n \in \mathbb{N}$, is an interpolating sequence for $H^\infty(\mathbb{D})$.
- Function spaces methods. In particular fundamental function of Köthe function space X

$$t \mapsto \phi(t) = \|\chi_{[0,t]}\|, \quad t \in \{\mu(A) : A \in \Sigma\}.$$

- Interpolation of operators. For all $p_0, p_1 \in (1, \infty)$, $p_0 \neq p_1$, $q_0, q_1, r \in [1, \infty]$, $s \in (0, 1)$,

$$(H^{p_0,q_0}, H^{p_1,q_1})_{s,r} = H^{p_s,r},$$

where $1/p_s = (1 - s)/p_0 + s/p_1$.

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