Weighted Extrapolation in Grand Lebesgue Spaces and Some Applications

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NPFSA 2017, Bedlewo, September 21, 2017

Abstract

We established weighted extrapolation results in grand Lebesgue spaces. In particular, we showed that if the one-weight inequality holds in the classical weighted Lebesgue space $L^{p_0}_w$ for a class of pairs of functions (f, g) and for all weights w from the Muckenhoupt class A_{p_0}, then the one-weight estimate also holds in grand Lebesgue spaces $L^{p),\theta}_w$ for the same pairs of functions (f, g) and for all Muckenhoupt weights $w \in A_p$. Our results cover both diagonal and off diagonal cases. Similar extrapolation results are obtained in weighted grand Lebesgue spaces defined with respect to product measure $\mu \times \nu$ on $X \times Y$, where (X, d, μ) and (Y, ρ, ν) are spaces of homogeneous type in terms of weights belonging to the Muckenhoupt-type classes defined with respect to products of balls. Based on these results we prove new one-weight estimates for Calderón–Zygmund and potential operators, and their commutators generally speaking, defined on spaces of homogeneous type (SHT).

Preliminaries

Let (X, d, μ) be a quasimetric measure space.
We assume that μ is a finite measure, $0 < \mu(B(x,r)) < \infty$ and $\mu\{x\} = 0$ for all $x \in X$ and $r > 0$.
If μ satisfies the doubling condition

$$\mu(B(x,2r)) \leq C\mu(B(x,r)), \tag{0.1}$$

with the positive constant C independent of $x \in X$ and $r > 0$, then (X, d, μ) is called a space of homogeneous type (SHT).

Introduction

Let (X, d, μ) be an SHT with finite measure and let $1 < p < \infty$, $\theta > 0$. Suppose that w is an a.e. positive integrable function on X. The weighted grand Lebesgue space $L^{p),\theta}_w(X)$ is the class of those $f : X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\theta}_w(X)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int\limits_X |f(x)|^{p-\varepsilon} w(x) d\mu(x) \right)^{1/(p-\varepsilon)}$$

is finite. If $w \equiv$ *const*, then we denote $L^{p),\theta}_w(X)$ by $L^{p),\theta}(X)$.

Preliminaries

The space $L^{p),\theta}(\Omega)$ for $\theta = 1$, where Ω is a bounded domain in $\mathbb{R}^n$ was introduced by T. Iwaniec and C. Sbordone in 1992 regarding the study of the integrability of the Jacobian under minimal hypothesis. This space for arbitrary positive θ was introduced by L. Greco, T. Iwaniec and C. Sbordone in 1997 when they studied solvability problems of certain PDEs.

Preliminaries

Together with $L_w^{p),\theta}(X,\mu)$ we are interested in the space $\mathcal{L}_w^{p),\theta}(X,\mu)$ which is defined by the norm $\|wf\|_{L^{p),\theta}(X,d\mu)}$, i.e.

$$\|f\|_{\mathcal{L}_w^{p),\theta}(X,\mu)} = \|wf\|_{L^{p),\theta}(X,d\mu)}.$$

Introduction

It is known that, there is a weight such that $\|w^{1/p} f\|_{L^{p)}(X)} \neq \|f\|_{L^{p)}_w(X)}$; thus, the weighted space $\mathcal{L}^{p)}_{w^{1/p}}(X)$ is different from the space $L^{p)}_w(X)$. The space $L^{p),\theta}_w(X)$ is a Banach space (see e.g. [Fiorenza]). It **is not** rearrangement invariant, separable, reflexive.
It is easy to see that the following continuous embeddings hold:

$$L^p_w(X) \hookrightarrow L^{p),\theta_1}_w(X) \hookrightarrow L^{p),\theta_2}_w(X) \hookrightarrow L^{p-\varepsilon}_w(X),$$

where $0 < \varepsilon \leq p-1$ and $\theta_1 < \theta_2$.

Muckenhoupt class of weights

Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt class $A_r(X)$ if

$$[w]_{A_r} := \sup_B \left(\frac{1}{\mu(B)} \int_B w d\mu \right) \left(\frac{1}{\mu(B)} \int_B w^{1-r'} d\mu \right)^{r-1} < \infty.$$

Let $1 < p, q < \infty$. Suppose that ρ is μ-a.e. positive function such that ρ^q is locally integrable. We say that $\rho \in \mathcal{A}_{p,q}(X)$ if

$$[\rho]_{\mathcal{A}_{p,q}(X)} := \sup_B \left(\frac{1}{\mu(B)} \int_B \rho^q d\mu \right) \left(\frac{1}{\mu(B)} \int_B \rho^{-p'} d\mu \right)^{q/p'} < \infty,$$

where the supremum is taken over all balls B in X.

Muckenhoupt class of weights

The one-weight characterization of various operators of Harmonic Analysis in Grand Lebesgue spaces is known under the Muckenhoupt condition on weights (see [Fiorenza, Gupta Jain, 2008], recent Monograph [Kokilashvili, Meskhi, Rafeiro, Samko, Birkhäuser-Springer, 2016] and references cited therein).

Extrapolation Result in the Classical Lebesgue Spaces

The next result is known as an extrapolation Theorem of Rubio De Francia.

Theorem A [Diagonal Case]. *Let* (X, d, μ) *be an SHT. Let for a family* $\mathcal{F}(X)$ *of pairs of nonnegative functions* (f, g), *for* $p_0 \in [1, \infty)$, *and for all* $w \in A_{p_0}(X)$ *we have*

$$\Big(\int_X g^{p_0} w d\mu \Big)^{\frac{1}{p_0}} \le C \Big(\int_X f^{p_0} w d\mu \Big)^{\frac{1}{p_0}}, \ \ (f, g) \in \mathcal{F},$$

where the positive constant C *does not depend on* (f, g), *and depends on* p_0 *and the characteristic* $\|w\|_{A_{p_0}}$. *Then for all* $1 < p < \infty$ *and all* $w \in A_p(X)$ *we have*

$$\Big(\int_X g^{p} w d\mu \Big)^{1/p} \le C_1 \Big(\int_X f^{p} w d\mu \Big)^{1/p}, \ \ (f, g) \in \mathcal{F},$$

with the positive constant C_1 *not depending on* (f, g).

Theorem B. [Off-diagonal Case]. *Let (X, d, μ) be an SHT. Let $1 \le p_0 < \infty$ and let $0 < q_0 < \infty$. Assume that for a family $\mathcal{F}(X)$ of pairs of nonnegative functions (f, g) and for all $w \in \mathcal{A}_{p_0,q_0}(X)$ we have*

$$\left(\int_X (gw)^{q_0} d\mu\right)^{\frac{1}{q_0}} \le C\left(\int_X (fw)^{p_0} w d\mu\right)^{\frac{1}{p_0}}, \quad (f, g) \in \mathcal{F},$$

where the positive constant C does not depend on (f, g). Then for all $1 < p < \infty$, $0 < q < \infty$ such that

$$\frac{1}{q_0} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p}$$

and all $w \in \mathcal{A}_{p,q}(X)$ we have

$$\left(\int_X (gw)^{q} d\mu\right)^{\frac{1}{q}} \le C_1\left(\int_X (fw)^{p} d\mu\right)^{\frac{1}{p}}, \quad (f, g) \in \mathcal{F},$$

with the positive constant C_1 independent on (f, g).

Extrapolation in grand Lebesgue spaces

Our results reed as follows (We formulate it for an SHT in the spirit of [Duoandikoetxea, 2011] which was derived for classical Lebesgue spaces):

Theorem. [Diagonal Case] Let (X, d, μ) be an SHT. Suppose that $p_0 \in [1, \infty)$. Let $\mathcal{F}(X)$ be the class of pairs of nonnegative functions defined on X. Suppose that for all $(f, g) \in \mathcal{F}(X)$ and for all $w \in A_{p_0}(X)$ we have

$$\left(\int\limits_X g^{p_0} w d\mu\right)^{\frac{1}{p_0}} \leq CN\big([w]_{A_{p_0}(X)}\big)\left(\int\limits_X f^{p_0} w d\mu\right)^{\frac{1}{p_0}},$$

where N is an increasing function and the constant c does not depend on w. Then for $1 < p < \infty$, $\theta > 0$, $w \in A_p(X)$ and $(f, g) \in \mathcal{F}$, we have

$$\|g\|_{L^{p),\theta}_w(X)} \leq \overline{C}\|f\|_{L^{p),\theta}_w(X)},$$

where $\overline{C}$ is the positive constant independent of $(f, g) \in \mathcal{F}(X)$.

Theorem. [Off-diagonal Case]. *Let* (X, d, μ) *be an SHT. Let* $1 < p_0 < \infty$ *and let* $1 < q_0 < \infty$. *Assume that* $\mathcal{F}(X)$ *is the class of pairs of nonnegative functions defined on* X. *Let for all* $(f, g) \in \mathcal{F}(X)$, *for* $w \in \mathcal{A}_{p_0,q_0}(X)$ *we have*

$$\left(\int_X (gw)^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq CN\left([w]_{\mathcal{A}_{p_0,q_0}(X)}\right) \left(\int_X (fw)^{p_0} d\mu \right)^{\frac{1}{p_0}},$$

where N *is an increasing function and the constant* C *does not depend on* w. *Then for* $1 < p < \infty$, $1 < q < \infty$ *such that*

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p},$$

for $\theta > 0$, $w \in \mathcal{A}_{p,q}(X)$ *and* $(f, g) \in \mathcal{F}$ *we have*

$$\|g\|_{\mathcal{L}_w^{q),\theta q/p}(X)} \leq \overline{C} \|f\|_{\mathcal{L}_w^{p),\theta}(X)},$$

where the positive constant $\overline{C}$ *is independent of* (f, g).

Applications in Calderón-Zygmund Theory

Let $k : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \le \frac{c}{\mu B(x, d(x,y))}, \quad x, y \in X, \quad x \neq y;$$

$$|k(x_1,y)-k(x_2,y)|+|k(y,x_1)-k(y,x_2)| \le c\omega\left(\frac{d(x_2,x_1)}{d(x_2,y)}\right)\frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) > d(x,x_2)$, where ω is a positive, non-decreasing function on $(0,\infty)$ satisfying Δ_2 condition ($\omega(2t) \le c\omega(t)$, $t > 0$) and the Dini condition $\int\limits_0^1 \omega(t)/t dt < \infty$.

We also assume that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}(X,\mu)$ the limit

$$(Tf)(x) = \lim_{\varepsilon \to 0} \int\limits_{X \setminus B(x,\varepsilon)} k(x,y) f(y) d\mu(y)$$

exists almost everywhere on X and that K is bounded in $L^{p_0}(X,\mu)$.

Applications: One-weight inequality

Let $b \in BMO(X)$, $m \in \mathbb{N} \cup \{0\}$ and let

$$T_b^m f(x) = \int_X [b(x) - b(y)]^m k(x,y) f(y) d\mu(y),$$

where k is the Calderón-Zygmund kernel.

Applications: One-weight inequality for commutators

Theorem *Let* $1 < p < \infty$ *and let* $\theta > 0$*. Suppose that Then there is a positive constant* C *such that for all* $f \in \mathcal{D}(X)$ *and all* $w \in A_p(X)$*,*

$$\|T_b^m f\|_{L_w^{p),\theta}(X)} \leq C\|b\|_{BMO(X)}^m \|M^{m+1} f\|_{L_w^{p),\theta}(X)}, \quad f \in \mathcal{D}(X),$$

where M^{m+1} *is the the Hardy–Littlewood maximal operator iterated* $m+1$ *times and* $\mathcal{D}(X)$ *is the class of all bounded functions on* X*.*

Applications: One-weight Inequality for Fractional Integrals

Let

$$I_\alpha f(x) = \int_X K_\alpha(x,y) f(y) d\mu(y), \quad x \in X,$$

where

$$K_\alpha(x,y) = \begin{cases} \mu(B_{xy})^{\alpha-1}, & x \neq y, \\ \mu\{x\}, & x = y,\ \mu\{x\} > 0, \end{cases}$$

$0 < \alpha < 1$ and $B_{xy} := B(x, d(x,y))$.
Suppose that

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y), \quad 0 < \alpha < 1.$$

Applications: One-weight Inequality for Fractional Integrals

Theorem. *Let* $1 < p < \infty$ *and let* $\theta > 0$*. Let* $w \in A_p(X)$*. Then there is a positive constant* C *such that*

$$\|I_\alpha f\|_{L_w^{p),\theta}(X)} \le C\|M_\alpha f\|_{L_w^{p),\theta}(X)}, \quad f \in \mathcal{D}(X)$$

Applications: One-weight inequality for commutators

$$I^m_{\alpha,b}f(x) = \int\limits_X [b(x) - b(y)]^m K_\alpha(x,y)d\mu(y), \ \ 0 < \alpha < 1,$$

$$\mathcal{I}^m_{\alpha,b}f(x) = \int\limits_X |b(x) - b(y)|^m K_\alpha(x,y)d\mu(y), \ \ 0 < \alpha < 1.$$

It is easy to see that, for $f \geq 0$, $|I^m_{\alpha,b}f(x)| \leq \mathcal{I}^m_{\alpha,b}f(x)$.
If $0 < p < \infty$, $0 < \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $w \in A_\infty(X)$, $b \in BMO(X)$, then there is a Constant C such that

$$\int\limits_X |\mathcal{I}^m_{\alpha,b}f(x)|^p w(x)d\mu(x) \leq C\|b\|^{mp}_{BMO(X)} \int\limits_X [M_\alpha(M^m f)(x)]^p w(x)d\mu(x).$$

Applications: One-weight inequality for commutators

Theorem *Let $1 < p < \infty$ and let $\theta > 0$. Suppose that $w \in A_p(X)$. Then there is a positive constant C such that*

$$\|\mathcal{I}^m_{\alpha,b} f\|_{L^{p),\theta}_w(X)} \leq C\|b\|^m_{BMO(X)}\|M_\alpha(M^m f)\|_{L^{p),\theta}_w(X)}, \quad f \in \mathcal{D}(X)$$

Product Spaces Case

Suppose that (X, d, μ) and (Y, ρ, ν) be spaces of homogeneous type with finite measure. Let $w(x, y)$ be a weight on $X \times Y$, i.e. w is a.e. positive integrable function on $X \times Y$.
Let $\mu(X) < \infty$, $\nu(X) < \infty$, $1 < p < \infty$ and let $\theta > 0$. The weighted grand Lebesgue space $L_w^{p),\theta}(X \times Y)$ is defined with respect to the norm

$$\|f\|_{L_w^{p),\theta}(X\times Y)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int\limits_{X\times Y} |f(x,y)|^{p-\varepsilon} w(x,y)\, d\mu \times \nu \right)^{1/(p-\varepsilon)}.$$

Since, in general, $\|w^{1/p} f\|_{L^{p,\theta}(\Omega)} \neq \|f\|_{L_w^{p,\theta}(\Omega)}$, we are also interested in the space $\mathcal{L}_w^{p),\theta}(X \times Y)$ which is defined by the norm
$\|f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)} = \|wf\|_{L^{p),\theta}(X\times Y)}$.

Product Spaces

Let $1 < r < \infty$. We say that a weight function w defined on $X \times Y$ belongs to the Muckenhoupt class $A_r^{(S)}$ if

$$[w]_{A_r^{(S)}} := \sup_{B_1 \times B_2} \left(\frac{1}{\mu(B_1)\nu(B_2)} \int\limits_{B_1 \times B_2} w \, d\mu \times \nu \right) \times$$

$$\left(\frac{1}{\mu(B_1)\nu(B_2)} \int\limits_{B_1 \times B_2} w^{1-r'} \, d\mu \times \nu \right)^{r-1} < \infty,$$

where the supremum is taken over all products of balls $B_1 \times B_2 \subset X \times Y$.

Product Spaces

Let $1 < p, q < \infty$. Suppose that ρ is μ-a.e. positive function such that ρ^q is locally integrable. We say that $\rho \in \mathcal{A}_{p,q}^{(S)}$ if

$$[\rho]_{\mathcal{A}_{p,q}^{(S)}} := \sup_{B_1 \times B_2} \left(\frac{1}{\mu(B_1)\nu(B_2)} \int\limits_{B_1 \times B_1} \rho^q \, d\mu \times \nu \right) \times$$

$$\left(\frac{1}{\mu(B_1)\nu(B_2)} \int\limits_{B} \rho^{-p'} \, d\mu \times \nu \right)^{q/p'} < \infty,$$

where the supremum is taken over all products of balls $B_1 \times B_2 \in X \times Y$. If $p = q$, then we denote $\mathcal{A}_{p,q}^{(S)}$ by $\mathcal{A}_p^{(S)}$.

Extrapolation for Product Case

Theorem (Diagonal Case)

Suppose that $p_0 \in [1, \infty)$. Let $\mathcal{F}(X \times Y)$ be the class of pairs of non-negative functions defined on $X \times Y$. Suppose that for all $(f, g) \in \mathcal{F}(X \times Y)$ and for all $w \in A_{p_0}^{(S)}$ the inequality

$$\left(\int_{X \times Y} g^{p_0} w \, d\mu \times \nu \right)^{\frac{1}{p_0}} \leq CN\left([w]_{A_{p_0}^{(S)}}\right) \left(\int_{X \times Y} f^{p_0} w \, d\mu \times \nu \right)^{\frac{1}{p_0}} \qquad (0.2)$$

holds, where N is a non-decreasing function and the constant C does not depend on w. Then for $1 < p < \infty$, $\theta > 0$, $w \in A_p^{(S)}$ we have

$$\|g\|_{L_w^{p),\theta}(X \times Y)} \leq \overline{C} \|f\|_{L_w^{p),\theta}(X \times Y)}, \quad (f, g) \in \mathcal{F}(X \times Y), \qquad (0.3)$$

where $\overline{C}$ is the positive constant independent of $(f, g) \in \mathcal{F}(X \times Y)$.

Extrapolation for Product Case

Theorem (Off-diagonal Case)

Let $1 < p_0 < \infty$ and let $1 < q_0 < \infty$. Assume that $\mathcal{F}(X \times Y)$ is the class of pairs of non-negative functions defined on $X \times Y$. Let for all $(f, g) \in \mathcal{F}(X \times Y)$, for $w \in \mathcal{A}^{(S)}_{p_0,q_0}$ we have

$$\left(\int\limits_{X\times Y} (gw)^{q_0} \, d\mu \times \nu \right)^{\frac{1}{q_0}} \leq CN\left([w]_{\mathcal{A}^{(S)}_{p_0,q_0}}\right) \left(\int\limits_{X\times Y} (fw)^{p_0} \, d\mu \times \nu \right)^{\frac{1}{p_0}}, \quad (0.4)$$

where N is a non-decreasing function and the constant C does not depend on w. Then for $1 < p < \infty$, $1 < q < \infty$ such that

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p},$$

for $\theta > 0$ and $w \in \mathcal{A}^{(S)}_{p,q}$ the inequality

Extrapolation for Product Case

$$\|g\|_{\mathcal{L}_w^{q),\theta q/p}(X\times Y)} \le \overline{C}\|f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)}, \quad (f,g)\in\mathcal{F}(X\times Y) \tag{0.5}$$

holds, where the positive constant $\overline{C}$ is independent of (f,g).

Applications

Let

$$M^{(S)}g(x,y) = \sup_{B_1 \ni x, B_2 \ni y} \frac{1}{(\mu B_1)(\nu B_2)} \int_{B_1 \times B_2} |g(t,s)| \, d\mu \times \nu(t,s)$$

be the strong fractional maximal operator on $(X \times Y)$.
Further, suppose that k_1 and k_2 are Calderón-Zygmund kernels defined on X and Y respectively. Let us introduce the notation:

$$E_{\varepsilon,\sigma}(x,y) := \Big(X \setminus B(x,\varepsilon)\Big) \times \Big(X \setminus B(y,\sigma)\Big).$$

Denote by Kf the multiple Calderón-Zygmund singular integral of f:

$$Kf(x,y) = \lim_{(\varepsilon,\sigma)\to(0,0)} \int_{E_{\varepsilon,\sigma}(x,y)} k_1(x,t)k_2(y,\tau)f(t,\tau) \, d\mu \times \nu(t,\tau).$$

Applications: double maximal and singular integrals

Theorem

Let $1 < p < \infty$ *and let* $\theta > 0$. *Then* $M^{(S)}$ *is bounded in* $\mathcal{L}_w^{p),\theta}(X \times Y)$ *if and only if* $w \in \mathcal{A}_p^{(S)}$.

Applications: double maximal and singular integrals

Theorem

Let $1 < p < \infty$ and let $\theta > 0$. Let $w \in \mathcal{A}_p^{(S)}$. Suppose that there is an exponent $r_0 > 1$ such that $Kf(x,y)$ exists for a.e. $(x,y) \in (X \times Y)$ when $f \in L^{r_0}(X \times Y)$ and, besides that, K it is bounded in $L^{r_0}(X \times Y)$. Then the following inequality

$$\|Kf\|_{\mathcal{L}_w^{p),\theta}(X\times Y)} \leq C\|f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)}$$

holds for all bounded f defined on $X \times Y$.

Before the boundedness of double maximal, Cauchy-type and other singular integral operators in grand Lebesgue spaces on product sets was studied by V. Kokilashvili (2010-2012).

Double fractional integral

Together with K we are interested in one-weight estimates for potential operator with product kernels (multiparameter fractional integral operator):

$$I_{\alpha_1,\alpha_2} f(x,y) = \int\limits_{X \times Y} \frac{f(t,\tau)\, d\mu \times \nu(t,\tau)}{\Big(\mu\big(B_1(x, d(x,t))\big)\Big)^{1-\alpha_1} \Big(\nu\big(B_2(y, \rho(y,\tau))\big)\Big)^{1-\alpha_2}},$$

$(x,y) \in X \times Y$, where $0 < \alpha_1, \alpha_2 < 1$.

Double fractional maximal operator

The appropriate strong maximal operator is given by the formula:

$$M^{(S)}_{\alpha_1,\alpha_2} f(x,y) = \sup_{(B_1 \times B_2 \ni (x,y)} \frac{1}{\mu(B_1)^{1-\alpha_1} \nu(B_2)^{1-\alpha_2}} \int_{B_1 \times B_2} |f(t,\tau)| dt d\tau.$$

Sobolev-type theorem for double potentials

Theorem

Let $1 < p < \infty$ *and let* $0 < \alpha < 1/p$*. We put* $q = \frac{p}{1-\alpha p}$*. Then the inequality*

$$\|I_{\alpha,\alpha} f\|_{\mathcal{L}_w^{q),\theta q/p}(X\times Y)} \leq C\|f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)}$$

holds for all $f \in \mathcal{L}_w^{p),\theta}(X \times Y)$ *if and only if* $w \in \mathcal{A}_{p,q}(X \times Y)$.

For the same result for double fractional integrals defined on $[0,1] \times [0,1]$ was derived by V. Kokilashvili and A.M., 2013 by using interpolation of operators in weighted Lebesgue spaces.

Theorem

Let $1 < p < \infty$ *and let* $\theta > 0$. *Let* $w \in A_p^{(S)}$ *(resp.* $w \in \mathcal{A}_p^{(S)}$*). Then there is a positive constant* C *such that*

$$\|I_{\alpha_1,\alpha_2} f\|_{L_w^{p),\theta}(X\times Y)} \leq C\|M_{\alpha_1,\alpha_2} f\|_{L_w^{p),\theta}(X\times Y)}$$

$\Bigg($ *resp.*

$$\|I_{\alpha_1,\alpha_2} f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)} \leq C\|M_{\alpha_1,\alpha_2} f\|_{\mathcal{L}_w^{p),\theta}(X\times Y)}.\Bigg)$$

References

The talk is based on the papers:

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