Polynomial inequalities via random processes and interpolation

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Outline

- Let $\mathcal{P}(\mathbb{C}^n)$ be the space of all polynomials $P\colon \mathbb{C}^n \to \mathbb{C}$,

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha(P) z^\alpha , \quad z = (z_1, ..., z_n) \in \mathbb{C}^n ,$$

where for a given multindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

denotes the αth monomial. As usual, $|\alpha| = \sum_{j=1}^n \alpha_j$ and we call

$$\deg(P) := \max\{|\alpha|\, ;\, c_\alpha(P) \neq 0\}$$

the total degree of P. If $m = \deg(P)$ and all monomial coefficients $c_\alpha = c_\alpha(P) = 0$ for all $|\alpha| < m$, then P is said to be m-homogeneous.

- Kahane-Salem-Zygmund (J. P. Kahane, 1993) For each positive integers m and n, there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$, $\varepsilon_\alpha = \pm 1$, such that

$$\sup_{z\in\mathbb{D}^n} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| \leqslant C n^{(m+1)/2} \sqrt{\log m}\,,$$

where C is a positive constant that depends neither on n nor on m.

(H. P. Boas, 2000) Let $1 \leqslant p \leqslant \infty$ and $m, n \geqslant 2$. Then there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$, $\varepsilon_\alpha = \pm 1$, such that

- If $1 \leqslant p \leqslant 2$, then

$$\sup_{z \in B_{\ell_p^{(n)}}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big| \leqslant C \sqrt{mn \log m}\, (m!)^{1-1/p}.$$

- If $2 \leqslant p \leqslant \infty$, then

$$\sup_{z \in B_{\ell_p^{(n)}}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big| \leqslant C \sqrt{mn \log m}\; n^{(1/2-1/p)m} (m!)^{1/2},$$

where C is a positive real number that depends neither on m nor on n.

- Let $1 \leqslant p \leqslant 2$ and $m, n \geqslant 2$. Then there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$ such that

$$\sup_{z \in B_{\ell_p^{(n)}}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha \Big| \leqslant C\sqrt{mn \log m}\,(m!)^{1-1/p} \sup_{|\alpha|=m} \Big\{ |c_\alpha| \frac{\alpha!}{m!} \Big\}.$$

- (A. Defant, D. Garcia and M. Maestre, 2003)

$$\sup_{z \in B_{\ell_p^{(n)}}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha \Big| \leqslant C^m n^{1-1/p} \sqrt{\log n} \sup_{|\alpha|=m} \Big\{ |c_\alpha| \sqrt{\frac{\alpha!}{m!}} \Big\},$$

where C is a positive real number that depends neither on m nor on n.

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- The set of all functions $\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which are non-decreasing in each variable and positively homogeneous (that is, $\varphi(\lambda s, \lambda t) = \lambda\varphi(s, t)$ for all $\lambda, s, t \geqslant 0$) is denoted by $\mathcal{U}$. For any $\varphi \in \mathcal{U}$, we define $\varphi_*\colon (0, \infty) \times (0, \infty) \to (0, \infty)$ by

$$\varphi_*(s, t) = 1/\varphi(s^{-1}, t^{-1}), \quad s, t > 0.$$

- The function ψ which corresponds to an exact positive interpolation functor $\mathcal{F}$ is given by

$$\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \psi(s, t)\mathbb{R}, \quad s, t > 0$$

  is called the characteristic function of the functor $\mathcal{F}$. Here $\alpha\mathbb{R}$ denotes $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\alpha\mathbb{R}} = \alpha|\cdot|$ for $\alpha > 0$.

- Definition Let (X_0, X_1) be a couple of Banach function lattices on a measure space (Ω, Σ, μ). For a given concave function $\psi \in \mathcal{U}$ $(\psi \in \widehat{\mathcal{U}})$ the Calderón-Lozanovsky space $\psi(X_0, X_1)$ consists of all $f \in L^0(\mu)$ such that

$$|f| \leqslant \lambda\, \psi(|f_0|, |f_1|) \quad \mu\text{-a.e.}$$

for some $f_j \in X_j$ with $\|f_j\|_{X_j} \leqslant 1$, $j = 0, 1$. Equipped with the norm

$$\|f\|_{\psi(X_0,X_1)} = \inf\left\{\lambda > 0;\ |f| \leqslant \lambda\, \psi(|f_0|, |f_1|),\ \|f_0\|_{X_0} \leqslant 1,\ \|f_1\|_{X_1} \leqslant 1\right\},$$

the space $\psi(X_0, X_1)$ forms a Banach function lattice.

- If $\Phi\colon [0,\infty) \to [0,\infty)$ is an Orlicz function (i.e., Φ is a convex increasing, function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$) and ψ is defined by

$$\psi(s,t) := t\Phi^{-1}(s/t), \quad s,t \geqslant 0, t > 0$$

and $\psi(0,0) = 0$. Then $\psi \in \mathcal{U}$, and, for any measure space (Ω, Σ, μ), the space $\psi(L_1, L_\infty)$ coincides isometrically with the Orlicz space $(L_\Phi, \|\cdot\|_\Phi)$,

$$L_\Phi := \left\{ f \in L^0(\mu);\, \Phi(|f|/\lambda) \in L_1(\mu) \right\},$$

where

$$\|f\|_\Phi = \inf\left\{\lambda > 0;\ \int_\Omega \Phi(|f|/\lambda)\, d\mu \leqslant 1\right\}.$$

- An Orlicz function φ is said to be an N-function, if

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

- Given a pseudo-metric (T, d), we denote by $N(T, d; \varepsilon)$ the entropy function associated with the pseudo-metric d on the set T for $\varepsilon > 0$, i.e.,

$$N(T, d; \varepsilon)$$

is the smallest number of open balls of radius $\varepsilon > 0$ in the pseudo-metric d needed to cover the set T.

- Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function. The entropy integral of (T, d) with respect to Φ is defined by

$$J_\Phi(T, d) = \int_0^{\Delta(T)} \Phi^{-1}\big(N(T, d; \varepsilon)\big)\, d\varepsilon,$$

where $\Delta(T) = \sup_{s,t \in T} d(s, t)$ denotes the diameter of T.

- If $(X_t)_{t\in T}$ is a stochastic process where T is an index set. Then

$$\mathbb{E}\big(\sup_{t\in T} X_t\big) := \sup\Big\{\mathbb{E}\big(\sup_{t\in F} X_t\big);\ F\subset T,\ F \text{ finite}\Big\},$$

where the right-hand side makes sense as soon as r.v. X_t is integrable for every $t\in T$.

- A fundamental example of stochastic processes is a random series,

$$X_t = \sum_k \xi_k f_k(t),$$

where (f_k) is a sequence of functions defined on a set T and (ξ_k) is a sequence of independent random variables on a measure space.

- The basis example is the random Fourier series,

$$X_t = \sum_k \xi_k e^{2\pi ikt}, \quad t\in[0,1].$$

- Pisier's Theorem If $(X_t)_{t\in T}$ is a stochastic process in the Orlicz space $L_\Phi(\Omega, \mathcal{A}, \mathbb{P})$ on a probability measure space such that

$$\|X_s - X_t\|_\Phi \leqslant d(s,t), \quad s,t \in T,$$

then we have

$$\mathbb{E}\big(\sup_{s,t\in T} |X_s - X_t|\big) \leqslant CJ_\Phi(T,d)$$

for some absolute constant $C > 0$.

Polynomial inequalities via random processes (M. M & R. Szwedek)

- For each m, $n \in \mathbb{N}$ the following index sets will be of special interest:

$$\mathcal{M}(m,n) = \{\mathbf{j} = (j_1, \ldots, j_m)\,;\, 1 \leqslant j_1, \ldots, j_m \leqslant n\} = \{1, \ldots, n\}^m,$$

$$\mathcal{J}(m,n) = \{\mathbf{j} = (j_1, \ldots, j_m) \in \mathcal{M}(m,n)\,;\, 1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_m \leqslant n\}.$$

- For $\mathbf{i}, \mathbf{j} \in \mathcal{M}(m,n)$, we write $\mathbf{i} \sim \mathbf{j}$ whenever there exists a permutation σ of $\{1, \ldots, m\}$ such that $(i_1, \ldots, i_m) = (j_{\sigma(1)}, \ldots, j_{\sigma(m)})$. Then, $\sim$ defines an equivalence relation on $\mathcal{M}(m,n)$. We denote by $[\mathbf{i}]$ the equivalence class of $\mathbf{i}$ and we put $|[\mathbf{i}]| := \mathrm{card}([\mathbf{i}])$.

- For every finite subset $\{x_1^*, \ldots, x_n^*\}$ in the dual X^* of a Banach space X and each $\mathbf{j} \in \mathcal{M}(m,n)$, an m-homogeneous polynomial $x_{\mathbf{j}}^* : X \to \mathbb{C}$ is defined by

$$x_{\mathbf{j}}^*(x) = x_{j_1}^*(x) \cdots x_{j_m}^*(x), \quad x \in X.$$

We note that $\mathbf{i} \sim \mathbf{j}$ implies that $x_{\mathbf{i}}^* = x_{\mathbf{j}}^*$, for each $\mathbf{i}, \mathbf{j} \in \mathcal{M}(m,n)$.

- Definition Given a Young function Φ and positive integers $m \geqslant 2$ and $n \geqslant 1$, Banach sequence spaces $E = E(\mathbb{N})$ and $F = F(\mathcal{J}(m,n))$ are said to be polynomially Φ-Bernoulli admissible with a constant $C(m)$, provided that for every Banach space X, every finite set of functionals $x_1^*, \ldots, x_n^* \in X^*$, every family $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ of independent random Bernoulli variables on a nonatomic probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and every sequence $(c_j)_{\mathbf{j}\in\mathcal{J}(m,n)}$ of complex numbers, we have

$$\int_\Omega \sup_{z\in B_X} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega)\, c_{\mathbf{j}}\, x_{\mathbf{j}}^*(z) \Big| \,\mathrm{d}\mathbb{P}(\omega)$$

$$\leqslant C(m) \big\| (c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)} \big\|_F \sup_{z\in B_X} \Big\| \sum_{k=1}^n x_k^*(z) e_k \Big\|_E^{m-1} J_\Phi(B_X, d),$$

where d is a pseudo-metric on B_X given by

$$d(z,w) = \Big\| \sum_{k=1}^n \big(x_k^*(z) - x_k^*(w)\big) e_k \Big\|_E, \quad (z,w) \in B_X \times B_X.$$

Lemma

Assume that X and $E^n = (\mathbb{C}^n, \|\cdot\|)$ are Banach spaces. For any set $\{x_1^*, \ldots, x_n^*\}$ of functionals in X^*, let d be a pseudo-metric on B_X given by

$$d(z,w) = \Big\| \sum_{k=1}^{n} \big(x_k^*(z) - x_k^*(w)\big) e_k \Big\|, \quad (z,w) \in B_X \times B_X.$$

Then, for every Orlicz function Φ we have

$$J_\Phi(B_X, d) \leqslant \Big(\sup_{z \in B_X} \sum_{k=1}^{n} |x_k^*(z)| \Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|).$$

- Definition Let Φ be an N-function and let L_Φ be the Orlicz space of functions on the nonatomic probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A Banach sequence space G modelled on a countable set I is said to be of Φ-Rademacher type if there is a constant $K > 0$ such that for every sequence (ε_i) of independent Bernoulli random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and every $(x_i) \in G$ we have

$$\Big\| \sum_{i\in I} \varepsilon_i\, x_i \Big\|_{L_\Phi(\mathbb{P})} \leqslant K\, \|(x_i)\|_G.$$

- Theorem Let $\mathcal{F}$ be an exact positive interpolation functor which is also a bounded lattice functor. If ψ is the characteristic function of $\mathcal{F}$, then

$$G(I) = \mathcal{F}(\ell_1(I), \ell_2(I))$$

is of Φ-Rademacher type with $\Phi(t) = e^{\Psi(t)} - 1$ for all $t \geqslant 0$, where the Orlicz function Ψ satisfies $\Psi^{-1}(t) \asymp \psi_*(1, \sqrt{t})$.

Theorem

Assume that $\psi \in \widehat{\mathcal{U}}$ is a super-multiplicative (i.e., $\psi(1,s)\psi(1,t) \leqslant \psi(1,st)$ for all $s,t > 0$) and satisfy $\psi(1,t) \to 0$ as $t \to 0+$ and $\psi(1,t) \to \infty$ as $t \to \infty$. Let $\mathcal{J} = \mathcal{J}(m,n)$ and $\mathcal{M} = \mathcal{M}(m,n)$ for each pair of integers $m \geqslant 2$, $n \geqslant 1$ and let $w = (w_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}}$, where $w_{\mathbf{j}} = \rho\big(|[\mathbf{j}]|, \sqrt{|[\mathbf{j}]|}\big)$, for each $\mathbf{j} \in \mathcal{J}$, and ρ is given by

$$\rho(s,t) := \inf_{u,v>0} \frac{\psi(su,tv)}{\psi(u,v)}, \quad s,t \geqslant 0.$$

Then the Banach spaces $E(\mathbb{N}) = \psi(\ell_1(\mathbb{N}), \ell_2(\mathbb{N}))$ and $F(\mathcal{J}) = \ell_\infty(1/w)$ are polynomially Φ-Bernoulli admissible with the constant $C(m) \leqslant Km$, where K is a universal constant and $\Phi(t) = e^{\Psi(t)} - 1$ for all $t \geqslant 0$ with Ψ an Orlicz function such that $\Psi^{-1}(t) \asymp \psi(1,\sqrt{t})$.

Theorem

Let $\mathcal{J} = \mathcal{J}(m, n)$ *for each pair of integers* $m \geqslant 2$, $n \geqslant 1$. *Suppose that* $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a concave and super-multiplicative function* $(\theta(s)\theta(t) \leqslant \theta(st)$ *for all* s, $t > 0)$. *Let* φ *be an Orlicz function given by* $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ *for all* $t > 0$. *Then the Banach sequence spaces*

$$\ell_\varphi(\mathbb{N}) \quad \text{and} \quad F(\mathcal{J}) = \ell_\infty(1/\varphi^{-1}(|[\mathbf{j}]|))$$

are polynomially Φ*-Bernoulli admissible with the constant* $C(m) \leqslant Km$ *for some* $K > 0$, *where* $\Phi(t) = e^{\Psi(t)} - 1$ *with* $\Psi^{-1}(t) = \theta(\sqrt{t})$ *for all* $t \geqslant 0$.

As a consequence of the preceding theorem we obtain the original result of Bayart (2012).

Theorem

Let $\mathcal{J} = \mathcal{J}(m,n)$ and $\mathcal{M} = \mathcal{M}(m,n)$ for each pair of integers $m \geqslant 2$, $n \geqslant 1$. If $1 < p \leqslant \infty$, then

$$\ell_p(\mathbb{N}) \quad \text{and} \quad F(\mathcal{J}) = \ell_\infty(|[\mathbf{j}]|^{-1/p})$$

are polynomially Φ-Bernoulli admissible Banach spaces with the constant $C(m) \leqslant Km$ for each $m \in \mathbb{N}$, where K is an absolute constant, and the Orlicz function $\Phi(t) = e^{t^q} - 1$ for all $t \geqslant 0$, where $1/p + 1/q = 1$.

Estimates of entropy integrals (M. M & R. Szwedek)

Theorem

Suppose that $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a concave and super-multiplicative function. Let* φ *be an Orlicz function given by* $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ *for all* $t > 0$. *Then there exists a constant* $K > 0$ *such that for every Banach space* X, *every finite set* $\{x_1^*, \ldots, x_n^*\}$ *of functionals in* X^*, *every family* $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ *of independent random Bernoulli variables on a nonatomic probability measure space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and every sequence* $(c_j)_{\mathbf{j}\in\mathcal{J}(m,n)}$ *of complex numbers, we have*

$$\int_\Omega \sup_{z\in B_X} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega) c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\Big| \,\mathrm{d}\mathbb{P}(\omega)$$

$$\leqslant Km \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)} \sup_{z\in B_X} \Big\| \sum_{k=1}^n x_k^*(z) e_k \Big\|_{\ell_\varphi}^{m-1} \Big(\sup_{z\in B_X} \sum_{k=1}^n |x_k^*(z)|\Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|_{\ell_\varphi^n}),$$

where $\Phi(t) = e^{\Psi(t)} - 1$ *with* $\Psi^{-1}(t) = \theta(\sqrt{t})$ *for all* $t \geqslant 0$.

Definition The characteristic function of a Banach space A with respect to (A_0, A_1) is defined by

$$\psi_A(s,t) = \sup\left\{\|a\|_A;\ a \in A_0 \cap A_1,\ \|a\|_{A_0} \leqslant s,\ \|a\|_{A_1} \leqslant t\right\}, \quad s,t > 0.$$

Theorem

Assume that ϕ, $\psi_E(1,\cdot)$ and an Orlicz function Φ and the space E are defined as in the above Theorem. If $\sup_{t\geqslant 1} \phi(t)\psi_E(1,1/t) < \infty$, then there exists a constant $C = C(\varphi,\psi) > 0$ such that the following estimate holds:

$$J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n}) \leqslant C\,\bar{\phi}(\log n)\log n, \quad n \geqslant 2,$$

where in what follows $\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)}$ for all $t \geqslant 0$.

Corollary

Let φ be an N-function and let Φ be an Orlicz function given by

$$\Phi^{-1}(t) \asymp \phi(\log(1+t)),$$

where $\phi(t) = t\varphi^{-1}(1/t)$ for all $t > 0$. Then then there exists a constant $C = C(\varphi) > 0$ such that for each $n \geqslant 2$ we have

$$J_\Phi(\ell_1^n, \|\cdot\|_{\ell_\varphi^n}) \leqslant C\,\bar{\phi}(\log n)\log n\,.$$

Unconditional basis constant of monomials and Bohr's radius (M. M & R. Szwedek)

- We denote by $\mathcal{P}(^mX)$ the Banach space of all m-homogenous scalar-valued polynomials P on the Banach space X, equipped with the norm

$$\|P\|_{\mathcal{P}(^mX)} = \sup\big\{|P(x)|;\ \|x\| \leqslant 1\big\}.$$

Given an n-dimension Banach space $X = (\mathbb{C}^n, \|\cdot\|)$, we exhibit some lower estimates of the unconditional basis constant for monomials z^α, $\alpha \in \mathbb{N}_0^n$, denoted by $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$.

- Recall that a basis (x_n) in a Banach space X is said to be unconditional if there exists $C \geqslant 1$ such that

$$\Big\|\sum_{k=1}^{n}\theta_k\lambda_k x_k\Big\| \leqslant C\Big\|\sum_{k=1}^{n}\lambda_k x_k\Big\|$$

for every choice of $(\lambda_k)_{k=1}^n$ and $(\theta_k)_{k=1}^n$ in $\mathbb{C}$ with $|\theta_k| = 1$; the smallest of the constants C is the unconditional basis constant of (x_n).

Let the canonical basis vectors $\{e_k\}_{k=1}^n$ form a normalized 1-unconditional basis in $X = (\mathbb{C}^n, \|\cdot\|)$.

- The Bohr radius of the open unit ball U_X of X, denoted by $K(U_X)$, is defined to be supremum over all $r \in [0,1]$ such that, when the power series $\sum_\alpha c_\alpha z^\alpha$ satisfies $\sup_{\|z\|<1} \left|\sum_\alpha c_\alpha z^\alpha\right| \leqslant 1$, then

$$\sup_{z \in rU_X} \sum_\alpha |c_\alpha z^\alpha| \leqslant 1.$$

- The multidimensional Bohr radius $K(\mathbb{D}^n)$ was introduced by H. P. Boas and D. Khavinson (1997) in the case $X := (\mathbb{C}^n, \|\cdot\|_\infty)$.
- H. Bohr proved that $K(\mathbb{D}) \geqslant 1/6$; the exact value $K(\mathbb{D}) = 1/3$ was obtained independently by M. Riesz, I. Schur and F. Wiener. Combining this result with the definitions gives (A. Defant, D. Garcia and M. Maestre, 2003),

$$\frac{1}{3}\,\frac{1}{\sup_m \chi_{\mathrm{mon}}\mathcal{P}(^m X)^{1/m}} \leqslant K(U_X) \leqslant \min\left\{\frac{1}{3}, \frac{1}{\sup_m \chi_{\mathrm{mon}}\mathcal{P}(^m X)^{1/m}}\right\}.$$

Theorem

Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space, and let φ be an Orlicz function given by $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ for all $t > 0$, where θ is a concave and super-multiplicative function. Then, for each $m \geqslant 2$ and $n \geqslant 2$, the following inequality holds true:

$$\frac{C\theta\big((m!)^{-1/2}\big)}{m\,\bar{\theta}\big(\sqrt{\log n}\big)\log n}\left(\frac{\sup_{z\in B_X}\sum_{k=1}^{n}|z_k|}{\sup_{z\in B_X}\|\sum_{k=1}^{n} z_k e_k\|_{\ell_\varphi}}\right)^{m-1} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}(^m X)\big),$$

where $\bar{\theta}(t) = \sup_{s>0}\frac{\theta(st)}{\theta(s)}$ for all $t > 0$.

Theorem

Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space, let $p \in (1,2]$ and let q be its conjugate exponent. Then, for each $m \geqslant 2$ and $n \geqslant 2$, the following inequality holds true:

$$\frac{C}{m\,(m!)^{1/q}\,(\log n)^{1+1/q}} \left(\frac{\sup_{z \in B_X} \sum_{k=1}^{n} |z_k|}{\sup_{z \in B_X} \|\sum_{k=1}^{n} z_k e_k\|_{\ell_p}} \right)^{m-1} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}(^m X)\big).$$

Corollary

Let φ be an Orlicz function given by $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ for all $t > 0$, where θ is a concave and super-multiplicative function. Then, for each $m \geqslant 2$ and $n \geqslant n$, the following inequality holds true:

$$\frac{C\theta\big((m!)^{-1/2}\big)\big(\theta(\sqrt{n})\big)^{m-1}}{m\,\bar{\theta}\big(\sqrt{\log n}\big)\log n} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}(^m \ell_\varphi^n)\big).$$