An Intersection Representation for Anisotropic Vector-valued Function Spaces

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New perspectives in the theory of function spaces and their applications

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Outline

1. Maximal L_q-L_p-Regularity

2. Anisotropic Mixed-norm Triebel-Lizorkin Spaces

3. An Intersection Representation in an Axiomatic Setting à la Hedberg&Netrusov

The L_q-L_p-maximal regularity space ($q, p \in (1, \infty)$)

A crucial aspect of the maximal L_q-L_p approach to

$$\partial_t u + \Delta u = f \text{ on } \mathscr{O}, \qquad u_{|\partial\mathscr{O}} = g \text{ on } \partial\mathscr{O}$$

is the spatial trace space of

$$W^{(2,1)}_{(p,q)}(\mathscr{O} \times J) = W^1_q(J; L_p(\mathscr{O})) \cap L_q(J; W^2_p(\mathscr{O})).$$

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Motivation for $p \neq q$:

- more nonlinearities can be treated.
- more interesting function space theory!

The Spatial Trace Space

On the one hand, (Weidemaier, 2002 ;DHP, 2007)

$$\mathrm{tr}_{\partial\mathscr{O}}\left[W^1_q(J; L_p(\mathscr{O})) \cap L_q(J; W^2_p(\mathscr{O}))\right] = F^{1-\frac{1}{2p}}_{q,p}(J; L_p(\partial\mathscr{O})) \cap L_q(J; F^{2-\frac{1}{p}}_{p,p}(\partial\mathscr{O})).$$

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On the other hand, (Johnsen & Sickel, 2008)

$$\mathrm{tr}_{\partial\mathscr{O}}\left[W^{(2,1)}_{(p,q)}(\mathscr{O}\times J)\right]=F^{1-\frac{1}{2p},(\frac{1}{2},1)}_{(p,q),p}(\partial\mathscr{O}\times J),$$

an anisotropic mixed-norm Triebel-Lizorkin space (see later).

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an anisotropic mixed-norm Triebel-Lizorkin space (see later).

So

$$F^{1-\frac{1}{2p},(\frac{1}{2},1)}_{(p,q),p}(\partial\mathscr{O}\times J)=F^{1-\frac{1}{2p}}_{q,p}(J;L_p(\partial\mathscr{O}))\cap L_q(J;F^{2-\frac{1}{p}}_{p,p}(\partial\mathscr{O})).$$

Goal

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This generality includes

$$F^{1-\frac{1}{2p},(\frac{1}{2},1)}_{(p,q),p}(\mathbb{R}^{n-1}\times\mathbb{R}) = F^{1-\frac{1}{2p}}_{q,p}(\mathbb{R};L_p(\mathbb{R}^{n-1})\cap L_q(\mathbb{R};F^{2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$$

and weighted extensions.

Anisotropic Mixed-norm Triebel-Lizorkin spaces

Let $\mathbf{p} \in (0,\infty)^d$, $q \in (0,\infty]$, $\mathbf{a} \in (0,\infty)^d$ and $s \in \mathbb{R}$.
The *anisotropic mixed-norm Triebel-Lizorkin space* $F^{s,\mathbf{a}}_{\vec{p},q}(\mathbb{R}^d)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|f\|_{F^{s,\mathbf{a}}_{\mathbf{p},q}(\mathbb{R}^d)} := \left\| \left\| (2^{sn} S_n f)_n \right\|_{\ell_q(\mathbb{N})} \right\|_{L_{\mathbf{p}}(\mathbb{R}^d)},$$

where $(S_n)_{n\in\mathbb{N}}$ is a Littlewood-Paley decomposition w.r.t. the **a**-anisotropic dilation

$$\delta^{\mathbf{a}}_{\lambda} x = (\lambda^{a_1} x_1, \ldots, \lambda^{a_d} x_d).$$

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Furthermore,

$$\|g\|_{L_{\mathbf{p}}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}} \left(\ldots \left(\int_{\mathbb{R}} |g(x)|^{p_1} dx_1 \right)^{p_2/p_1} \ldots \right)^{1/p_d} dx_d \right)^{p_d}.$$

An Intersection Representation

Theorem (Denk & Kaip, 2013)

Let $p, q \in (1, \infty)$, $s > 0$, $a, b \in (0, \infty)$.
For

$$\mathbf{p} := (p, \dots, p, q) \in (1, \infty)^d \qquad \text{and} \qquad \mathbf{a} := (a, \dots, a, b) \in (0, \infty)^d$$

it holds that

$$F^{s,\mathbf{a}}_{\mathbf{p},p}(\mathbb{R}^d) = F^{s/b}_{q,p}(\mathbb{R}; L_p(\mathbb{R}^{d-1})) \cap L_q(\mathbb{R}; F^{s/a}_{p,p}(\mathbb{R}^{d-1}))$$

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An intersection representation in weighted setting: L., master thesis, 2014. But more general version to be stated later.

The Fubini Property

Theorem (The Fubini Property (Kaljabin 1980; Triebel, 1999))

Let $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in \mathbb{R}$.
If $s > d(\frac{1}{p} - 1, \frac{1}{q} - 1)_+$, then

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d)} \eqsim \sum_{j=1}^{d} \Big\| \|f\|_{F^s_{p,q}(\mathbb{R})} \Big\|_{L_p(\mathbb{R}^{d-1})}, \qquad f \in F^s_{p,q}(\mathbb{R}^d).$$

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Dachkovski, 2003: Fubini property for $B^{s,\mathbf{a}}_{p,p}(\mathbb{R}^d)(= F^{s,\mathbf{a}}_{p,p}(\mathbb{R}^d))$.

The Setting I/III

Let

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_l}, \qquad \mathbf{d} = (d_1, \ldots, d_l) \in (\mathbb{Z}_{>0})^l, l \in \mathbb{Z}_{>0},$$

with **d**-anisotropy $\mathbf{A} = (A_1, \ldots, A_l)$; each A_j is a real $d_j \times d_j$ matrix with $\sigma(A_j) \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$.

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$\mathbf{A}$ gives rise to a one-parameter group of expansive dilations $(\mathbf{A}_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}^d$:

$$\mathbf{A}_t x = (\exp[A_1 \ln(t)]x_1, \ldots, \exp[A_d \ln(t)]x_d)\,.$$

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Example: $\mathbf{A} = (a_1 I_{d_1}, \ldots, a_d I_d)$, where $a_1, \ldots, a_d \in (0, \infty)$, has corresponding dilations

$$\mathbf{A}_t x = (t^{a_1} x_1, \ldots, t^{a_d} x_d).$$

The Setting II/III

Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a $\mathbf{d}$-anisotropy of $\mathbb{R}^d$ and $(S, \mathscr{A}, \mu)$ a σ-finite measure space.

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$\mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, \mathbf{r}, (S, \mathscr{A}, \mu)) :=$ the set of all quasi-Banach function spaces E on $\mathbb{R}^d \times \mathbb{N} \times S$ with the Fatou property s.t.:

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(a) [shifts] $S_+, S_- \in \mathcal{B}(E)$, the left respectively right shift on $\mathbb{N}$, with

$$\left\|(S_+)^n\right\|_{\mathcal{B}(E)} \lesssim 2^{-\varepsilon_+ n} \quad \text{and} \quad \left\|(S_-)^n\right\|_{\mathcal{B}(E)} \lesssim 2^{\varepsilon_- n}, \qquad n \in \mathbb{N}.$$

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(b) [maximal function] $M^{\mathbf{A}}_{\mathbf{r}} = M_{A_d, r_d} \dots M_{A_1, r_1}$ is bounded on E, where

$$M_{A_j, r_j} f(x) = \sup_{\delta > 0} \left(\fint_{B^{A_j}(x_j, \delta)} |f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)|^{r_j} dy \right)^{1/r_j}.$$

The Setting III/III

Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, \mathbf{r}, (S, \mathscr{A}, \mu))$.
$Y^{\mathbf{A}}(E) :=$ the space of all $f \in L_0(S; \mathcal{S}'(\mathbb{R}^d))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; \mathcal{S}'(\mathbb{R}^d))$$

with $(f_n)_n \subset L_0(S; \mathcal{S}'(\mathbb{R}^d))$ satisfying the Fourier support condition

$$\operatorname{supp} \hat{f}_0 \subset B^{\mathbf{A}}(0,2)$$
$$\operatorname{supp} \hat{f}_n \subset B^{\mathbf{A}}(0,2^{n+1}) \setminus B^{\mathbf{A}}(0,2^{n-1}), \qquad n \in \mathbb{N},$$

and $(f_n)_n \in E$, equipped with the quasinorm

$$\|f\|_{Y^{\mathbf{A}}(E)} := \inf \left\|(f_n)\right\|_E .$$

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Example (with trivial $S = \{0\}$):

$$\left. \begin{array}{l} \mathbf{A} = (a_1 I_1, \ldots, a_d I_1), \mathbf{a} \in (0,\infty)^d \\ E = L_{\mathbf{p}}(\mathbb{R}^d)[\ell^s_q(\mathbb{N})] \end{array} \right\} \rightsquigarrow Y^{\mathbf{A}}(E) = F^{s,\mathbf{a}}_{\mathbf{p},q}(\mathbb{R}^d).$$

The Main Result

Let E be a quasi-Banach function space on $\mathbb{R}^d \times \mathbb{N}$.
For each $j \in \{1, \dots, l\}$, let $E_{[\mathbf{d};j]}$ be E viewed a qBFS on $\mathbb{R}^{d_j} \times \mathbb{N} \times \mathbb{R}^{d-d_j}$.
Note:

$$E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, \mathbf{r}) \Longrightarrow E_{[\mathbf{d};j]} \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A_j, r_j, \mathbb{R}^{d-d_j}).$$

Theorem (L., 2017)

Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, \mathbf{r})$. If $\varepsilon_+ > \sum_{j=1}^{l} \mathrm{tr}(A_j)(\frac{1}{r_j} - 1)_+$, then

$$Y^{\mathbf{A}}(E) = \bigcap_{j=1}^{l} Y^{A_j}(E_{[\mathbf{d};j]})$$

with an equivalence of quasi-norms.

The Main Result with an Example

Theorem (L., 2017)

Let $E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, \mathbf{r})$*. If* $\varepsilon_+ > \sum_{j=1}^{l} \operatorname{tr}(A_j)(\frac{1}{r_j} - 1)_+$*, then*

$$Y^{\mathbf{A}}(E) = \bigcap_{j=1}^{l} Y^{A_j}(E_{[\mathbf{d};j]})$$

with an equivalence of quasi-norms.

Example: $\mathbf{p} \in (0,\infty)^l$, $q \in (0,\infty]$, $\mathbf{w} \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j}, A_j)$, $s \in \mathbb{R}$ gives

$$F^{s,\mathbf{A}}_{\mathbf{p},q}(\mathbb{R}^d, \mathbf{w}; X) = \bigcap_{j=1}^{l} F^{s,[\mathbf{d};j],A_j}_{\mathbf{p},q}(\mathbb{R}^d, \mathbf{w}; X)$$

for appropriate s.