Factorization of the identity in SL^∞ and mixed-norm Hardy spaces

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Overview

1. A description of the problem class
2. Results in one-parameter functions spaces
3. Results in bi-parameter function spaces
4. Local results

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A description of the problem class

- X a Banach space, $T : X \to X$ a bounded operator.
- Find conditions on X and T such that the identity Id on X factors through T i.e.

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}} & X \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ X & \xrightarrow[T]{} & X \end{array} \qquad \|R\|\|S\| \le C.$$

- The problem has finite dimensional (quantitative) and infinite dimensional (qualitative) aspects.
- Classical examples for X include: ℓ^p (Pełczyński), L^1 (Enflo–Starbird), L^p (Andrew/Johnson–Maurey–Schechtman–Tzafriri), ℓ^p_n (Bourgain–Tzafriri), $L^p(L^q)$, $1 < p, q < \infty$ (Capon).

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Large diagonal

- X a Banach space, $T : X \to X$ a bounded operator.
- X has an unconditional basis (b_n), and let $b_n^* \in X^*$ be so that $\langle b_n, b_m^* \rangle = 0$, $m \neq n$, and $\langle b_n, b_n^* \rangle = 1$.
- T has large diagonal (relative to (b_n)) if $\inf_{n \in \mathbb{N}} |\langle T b_n, b_n^* \rangle| > 0$.
- Then for many Banach spaces X we know that

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Theorem (N. J. Laustsen, R. L., P. F. X. Müller)

There is a bounded operator T on a Banach space X with an unconditional basis such that T has large diagonal, but the identity operator on X does not factor through T.

- X is Gowers' space with an unconditional basis (Gowers–Maurey).

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Defining SL^∞

- $\mathcal{D} = \{[\frac{k-1}{2^n}, \frac{k}{2^n}) : n \geq 0,\ 1 \leq k \leq 2^n\}$ denotes the dyadic intervals,
- h_I the L^∞-normalized Haar function supported on $I \in \mathcal{D}$.
- $f = \sum_{I\in\mathcal{D}} a_I h_I$ is a formal series,
- $\mathbb{S}(f) = \big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\big)^{1/2}$ is the dyadic square function of f.
- $SL^\infty = \{f = \sum_{I\in\mathcal{D}} a_I h_I : \|f\|_{SL^\infty} < \infty\}$ equipped with the norm

$$\|f\|_{SL^\infty} = \|\,\mathbb{S}(f)\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in[0,1)} \Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2(x)\Big)^{1/2},$$

  is a non-separable Banach space.
- If $f = \sum_{I\in\mathcal{D}} a_I h_I \in SL^\infty$, then $\langle f, h_I\rangle = a_I|I|$, $I \in \mathcal{D}$.

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Operators with large diagonal in SL^∞

Theorem (R. L.)

Let $\delta > 0$, and let $T : SL^\infty \to SL^\infty$ be a bounded operator with large diagonal, i.e. satisfying

$$|\langle Th_I, h_I\rangle| \geq \delta |I|, \qquad I \in \mathcal{D}.$$

Then there exist bounded operators $R, S : SL^\infty \to SL^\infty$ such that

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\mathrm{Id}} & SL^\infty \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ SL^\infty & \xrightarrow[T]{} & SL^\infty \end{array} \qquad \|R\|\|S\| \leq \frac{1}{\delta} + \frac{1}{10000}. \tag{2.1}$$

- Nature of the proof: we use infinite dimensional methods directly in the non-spearable space SL^∞ (specifically, we do not use Bourgain's localization method).

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A Banach space X is primary if for every bounded projection $Q : X \to X$, either $Q(X)$ or $(\mathrm{Id} - Q)(X)$ is isomorphic to X.

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The Gamlen-Gaudet construction (1973)

- The elementary building blocks for R, S are b_I, $I \in \mathcal{D}$. They are created by the Gamlen-Gaudet construction:

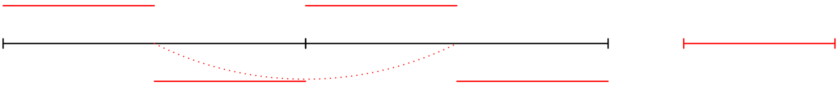

Figure: On the left side: construction of $b_I = \sum_{K \in \mathcal{B}_I} h_K$. On the right side: the corresponding index intervals I.

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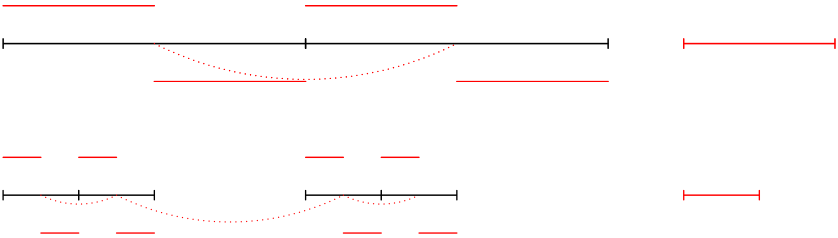

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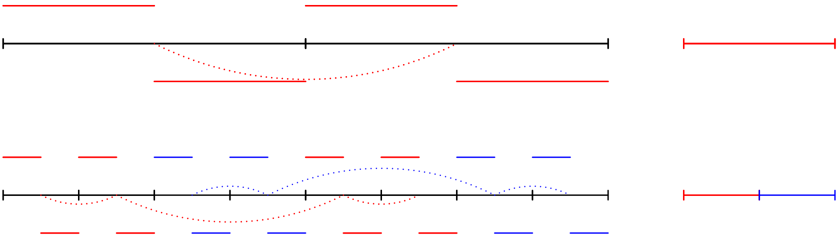

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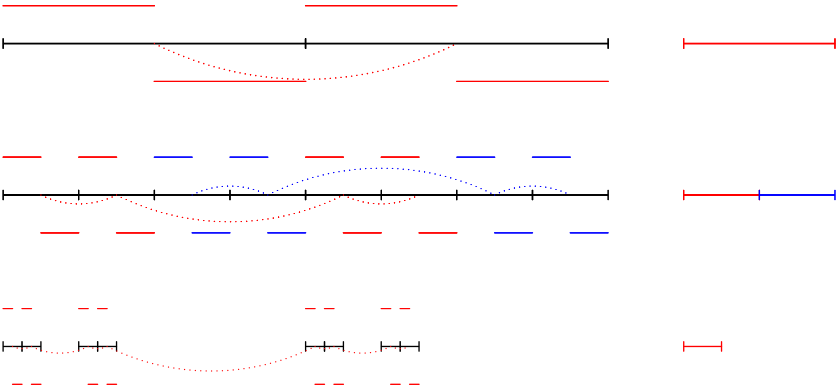

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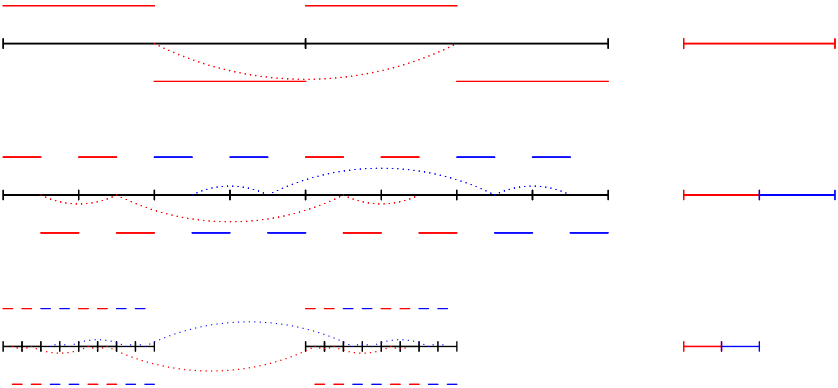

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Overview

Mixed-norm Hardy spaces $H^p(H^q)$

- $\mathcal{R} = \{I \times J : I, J \in \mathcal{D}\}$ denotes the dyadic rectangles on the unit square,
- $h_{I\times J}(x, y) = h_I(x)h_J(y)$ the L^∞-normalized tensor product Haar function, $I \times J \in \mathcal{R}$.
- $f = \sum_{I\times J\in\mathcal{R}} a_{I\times J}h_{I\times J}$ a formal series,
- $\mathbb{S}(f) = \big(\sum_{I\times J\in\mathcal{R}} a_{I\times J}^2 h_{I\times J}^2\big)^{1/2}$ the square function of f.
- Given $1 \leq p, q < \infty$, we define the mixed norm Hardy space

$$H^p(H^q) = \Big\{f = \sum_{I\times J\in\mathcal{R}} a_{I\times J}h_{I\times J} : \|f\|_{H^p(H^q)} < \infty\Big\},$$

where

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$$H^p(H^q) = \Big\{f = \sum_{I\times J\in\mathcal{R}} a_{I\times J} h_{I\times J} : \|f\|_{H^p(H^q)} < \infty\Big\},$$

where

$$\begin{aligned}\|f\|_{H^p(H^q)} &= \|\mathbb{S}(f)\|_{L^p(L^q)}\\ &= \Big(\int_0^1 \Big(\int_0^1 \big(\sum_{R\in\mathcal{R}} a_R^2 h_R^2(x,y)\big)^{q/2} \,\mathrm{d}y\Big)^{p/q} \,\mathrm{d}x\Big)^{1/p}.\end{aligned}$$

Mixed-norm Hardy spaces $H^p(H^q)$

- $\mathcal{R} = \{I \times J : I, J \in \mathcal{D}\}$ denotes the dyadic rectangles on the unit square,
- $h_{I\times J}(x,y) = h_I(x)h_J(y)$ the L^∞-normalized tensor product Haar function, $I \times J \in \mathcal{R}$.
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Large diagonal relative to the bi-parameter Haar system

Theorem (N. J. Laustsen, R. L., P. F. X. Müller)

Let $1 \leq p, q < \infty$, $\delta > 0$ *and* $T : H^p(H^q) \to H^p(H^q)$ *be a bounded operator with large diagonal, i.e.*

$$|\langle Th_{I\times J}, h_{I\times J}|\rangle \geq \delta|I \times J|, \qquad I \times J \in \mathfrak{R}.$$

Then there are bounded operators $R, S : H^p(H^q) \to H^p(H^q)$ *such that*

$$\begin{array}{ccc} H^p(H^q) & \xrightarrow{\text{Id}} & H^p(H^q) \\ R\downarrow & & \uparrow S \\ H^p(H^q) & \xrightarrow[T]{} & H^p(H^q) \end{array} \qquad \|R\|\|S\| \leq C/\delta,$$

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What the Basic building blocks $b_{I\times J}$ for R and S in $H^p(H^q)$ look like

- $\mathcal{X}_{I\times J}, \mathcal{Y}_{I\times J} \subset \mathcal{D}$, finite, non-empty, $\quad I \times J \in \mathcal{R}$;
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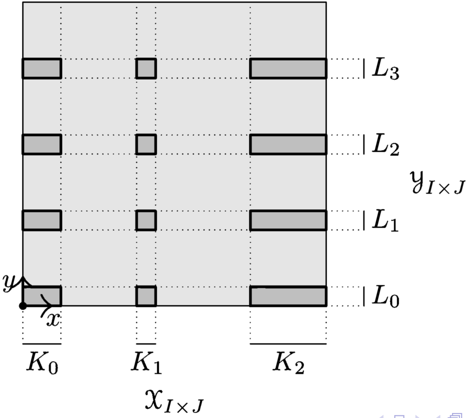

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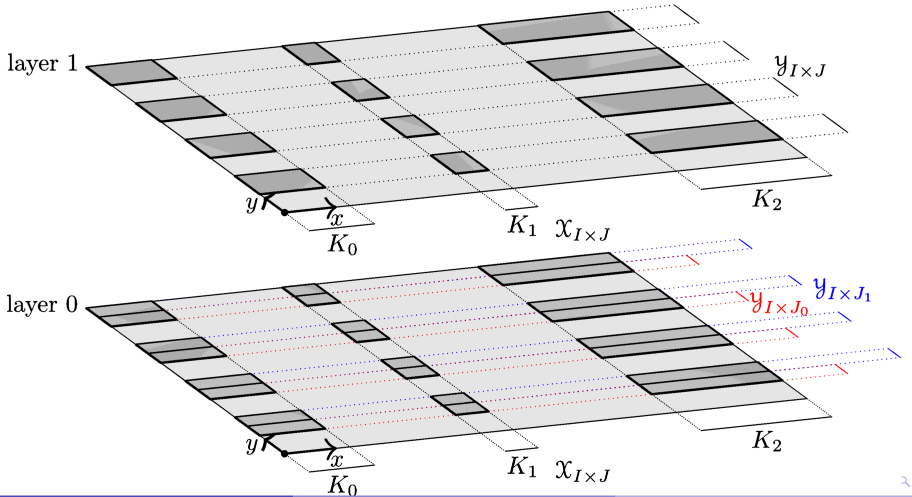

Overview

1. A description of the problem class
2. Results in one-parameter functions spaces
3. Results in bi-parameter function spaces
4. Local results

Local results

X_n can be any of the following spaces (no mixing allowed!):

- $\mathrm{BMO}_n(\mathrm{BMO}_n)$ (R. L.–P. F. X. Müller),
- $H^p_n(H^q_n)$, $1 \le p, q < \infty$ (R. L.),
- $H^s_n(\mathrm{BMO}_n)$, $\mathrm{BMO}_n(H^s_n)$, $1 < s < \infty$, (R. L.),
- SL^∞_n (R. L.).

Theorem

Given $\Gamma > 0$ and $n \in \mathbb{N}$, there exists an integer $N = N(\Gamma, n)$ such that for any $T : X_N \to X_N$ with $\|T\| \le \Gamma$ we can find bounded operators R_n, S_n such that

$$\begin{array}{ccc} X_n & \xrightarrow{\mathrm{Id}} & X_n \\ {\scriptstyle R_n}\downarrow & & \uparrow{\scriptstyle S_n} \\ X_N & \xrightarrow[H]{} & X_N \end{array} \qquad \|R_n\|\|S_n\| \le 2 + \frac{1}{10000},$$

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