Compactness criteria in $L^p$, $p > 0$

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Historically first compactness criterion in function space was a following statement, giving condition of completely boundedness in the space of continuous function.

**Theorem (C.Arzela–G.Ascoli criterion)**

Let $X$ be compact metric space. The set $S \subset C(X)$ is completely bounded if and only if $S$ is uniformly bounded and equicontinuous, that is

\[ \exists M > 0 \quad \forall f \in S, \ x \in X \quad |f(x)| \leq M \]

\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in S \quad \forall x_1, x_2 \in X \]

\[ d(x_1, x_2) < \delta \quad \implies \quad |f(x_1) - f(x_2)| < \varepsilon. \]
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Compactness in $L^p(X)$
Notations

Let \((X, d, \mu)\) be bounded metric space with metric \(d\) and Borel measure \(\mu\),

\[ B = B(x, r) = \{ y \in X : d(x, y) < r \}, \]

\(r_B\) is the radius of \(B\),

\[ f_B = \int_B f \, d\mu = \frac{1}{\mu_B} \int_B f \, d\mu \]

Doubling condition

\[ \mu_B(x, 2r) \leq c_\mu \mu_B(x, r), \quad x \in X, \quad r > 0. \quad (1) \]

\[ \| f \|_{L^p} = \| f \|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}, \quad p > 0. \]
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Theorem (A.N.Kolmogorov criterion)

\[ S \subset L^p(X), \ p \geq 1, \text{ is completely bounded if and only if } S \text{ is bounded and } \]
\[ \lim_{r \to +0} \sup_{f \in S} \int_X \left| f(x) - \int_{B(x,r)} f \, d\mu \right|^p \, d\mu(x) = 0. \]

A.N.Kolmogorov (1931), \( X \subset \mathbb{R}^n \) is bounded and measurable (all functions are zero outside of \( X \)).
A.Kałamajska (1999), on metric measure spaces with the property
\[ \forall \ r > 0 \quad \inf_{x \in X} \mu B(x, r) > 0. \]
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Luzin theorem and compactness
Let $\Omega$ be the class of functions $\eta : (0, 1] \to \mathbb{R}_+$,

$$\eta(+0) = 0, \quad \eta(t) \uparrow.$$ 

Let $D_\eta(f)$ be the set of functions $0 \leq g \in L^0(X)$ with

$$\exists E \subset X \quad \mu E = 0$$

$$|f(x) - f(y)| \leq [g(x) + g(y)]\eta(d(x, y)), \quad x, y \in X \setminus E. \quad (2)$$

This inequality (2) is called local smoothness inequality.
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Main point of new criterion of compactness is the following: the function \( \eta \in \Omega \) in local smoothness inequality is the same for whole compact, and functions \( g \) from this inequality satisfy some uniform estimate.

**Theorem**

The set \( S \subset L^p(X) \), \( p > 0 \), is completely bounded if and only if \( S \) is bounded and

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\exists \eta \in \Omega \quad \sup_{f \in S} \inf_{g \in D_\eta(f)} \|g\|_{L^p(X)} < +\infty
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If we replace here \( L^p \) by \( C \), then we obtain perfectly Arzela–Ascoli criterion for \( C(X) \).
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Maximal functions and compactness
The construction of functions $g$ from local smoothness inequality goes back to works of A.Calderon, K.I.Oskolkov and V.I.Kolyada.

For $q > 0$ and $\eta \in \Omega$ denote by

$$
N^q_{\eta} f(x) = \sup_{B \ni x} \frac{1}{\eta(r_B)} \left( \int_B |f(x) - f(y)|^q d\mu(y) \right)^{1/q}.
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The main property now is the local smoothness type inequality

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|f(x) - f(y)| \leq c_q [N^q_{\eta} f(x) + N^q_{\eta} f(x)] \eta(d(x, y)).
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$$|f(x) - f(y)| \leq c_q \left[ \mathcal{N}_q^\eta f(x) + \mathcal{N}_q^\eta f(x) \right] \eta(d(x, y)).$$
Theorem

Let $S \subset L^p(X)$, $p > 0$, be bounded set. Then

1) if $0 < q < p$ and $S$ is completely bounded, then

$$\exists \eta \in \Omega \sup_{f \in S} \|N^q_{\eta}f\|_{L^p(X)} < +\infty,$$

(3)

2) if for some $q > 0$ the condition (3) is fulfilled then $S$ is completely bounded.

The first statement of this theorem is not true for $q = p$.

The second part this theorem can be strengthened, and the condition (3) can be weakened.
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New conditions for compactness
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Let $B \subset X$ be a ball. Then there exists the number $I_B^{(p)} f$, such that

$$\int_B |f - I_B^{(p)} f|^p d\mu = \inf_{c \in \mathbb{R}} \int_B |f - c|^p d\mu.$$ 

Theorem

Let $S \subset L^p(X)$, $p > 0$, be bounded set. If for some $q > 0$ the following condition holds

$$\lim_{r \to +0} \sup_{f \in S} \int_X \left[ \int_{B(x,r)} |I_{B(x,r)}^{(q)} f - f(y)|^q d\mu(y) \right]^{p/q} d\mu(x) = 0 \quad (5)$$

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The last two theorems looks very similar. But the proofs are quite different.
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It is natural to consider the following condition

$$\lim_{r \to 0} \sup_{f \in S} \int_{X} |I_{B(x,r)}^{(q)} f - f(x)|^p \, d\mu(x) = 0.$$  \hspace{1cm} (6)

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Thank you for attention!