Compactness criteria in L^p, $p > 0$

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Compactness in C(X)

Historically first compactness criterion in function space was a following statement, giving condition of completely boundedness in the space of continuous function.

Theorem (C.Arzela–G.Ascoli criterion)

Let X be compact metric space. The set $S \subset C(X)$ is completely bounded if and only if S is uniformly bounded and equicontinuous, that is

$$\exists M > 0 \quad \forall f \in S,\ x \in X \quad |f(x)| \leqslant M$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in S \quad \forall x_1, x_2 \in X$$

$$d(x_1, x_2) < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon.$$

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Compactness in $L^p(X)$

Notations

Let (X, d, μ) be bounded metric space with metric d and Borel measure μ,

$$B = B(x, r) = \{y \in X : d(x, y) < r\},$$

r_B is the radius of B,

$$f_B = \fint_B f d\mu = \frac{1}{\mu B} \int_B f d\mu$$

Doubling condition

$$\mu B(x, 2r) \leqslant c_\mu \mu B(x, r), \quad x \in X, \quad r > 0. \tag{1}$$

$$\|f\|_{L^p} = \|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}, \quad p > 0.$$

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Theorem (A.N.Kolmogorov criterion)

$S \subset L^p(X)$, $p \geqslant 1$, is completely bounded if and only if S is bounded and

$$\lim_{r \to +0} \sup_{f \in S} \int_X \left| f(x) - \fint_{B(x,r)} f \, d\mu \right|^p d\mu(x) = 0.$$

A.N.Kolmogorov (1931), $X \subset \mathbb{R}^n$ is bounded and measurable (all functions are zero outside of X).
A.Kałamajska (1999), on metric measure spaces with the property

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Luzin theorem and compactness

Notations

Let Ω be the class of functions $\eta : (0, 1] \to \mathbb{R}_+$,

$$\eta(+0) = 0, \quad \eta(t) \uparrow .$$

Let $D_\eta(f)$ be the set of functions $0 \leqslant g \in L^0(X)$ with

$$\exists\, E \subset X \quad \mu E = 0$$

$$|f(x) - f(y)| \leqslant [g(x) + g(y)]\eta(d(x, y)), \quad x, y \in X \setminus E. \tag{2}$$

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Main point of new criterion of compactness is the following: the function $\eta \in \Omega$ in local smoothness inequality is the same for whole compact, and functions g from this inequality satisfy some uniform estimate.

Theorem

The set $S \subset L^p(X)$, $p > 0$, is completely bounded if and only if S is bounded and

$$\exists\, \eta \in \Omega \quad \sup_{f \in S} \inf_{g \in D_\eta(f)} \|g\|_{L^p(X)} < +\infty$$

If we replace here L^p by C, then we obtain perfectly Arzela–Ascoli criterion for $C(X)$.

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Maximal functions and compactness

The construction of functions g from local smoothness inequality goes back to works of A.Calderon, K.I.Oskolkov and V.I.Kolyada.

For $q > 0$ and $\eta \in \Omega$ denote by

$$\mathcal{N}_\eta^q f(x) = \sup_{B \ni x} \frac{1}{\eta(r_B)} \left(\fint_B |f(x) - f(y)|^q d\mu(y) \right)^{1/q}.$$

$X = \mathbb{R}^n$, $\eta(t) = t^\alpha$ A.Cálderon (1972), and later A.Cálderon–R.Scott (1978),
$X = [0,1]$, $\eta(t)t^{-1} \downarrow$ K.I.Oskolkov (1977, in implicit form),
$X = [0,1]^n$, $\eta(t)t^{-1} \downarrow$ V.I.Kolyada (1987), extended Oskolkov results on functions of several variables, in general case I.A.Ivanishko (2004).
The main property now is the local smoothness type inequality

$$|f(x) - f(y)| \leqslant c_q[\mathcal{N}_\eta^q f(x) + \mathcal{N}_\eta^q f(x)]\eta(d(x,y)).$$

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Theorem

Let $S \subset L^p(X)$, $p > 0$, be bounded set. Then
1) if $0 < q < p$ and S is completely bounded, then

$$\exists\, \eta \in \Omega \quad \sup_{f \in S} \left\| \mathcal{N}_\eta^q f \right\|_{L^p(X)} < +\infty, \tag{3}$$

2) if for some $q > 0$ the condition (3) is fulfield then S is completely bounded.

The first statement of this theorem is not true for $q = p$.
The second part this theorem can be strengthened, and the condition (3) can be weakened.

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then S *is completely bounded.*

New conditions for compactness

Let $B \subset X$ be a ball. Then there exists the number $I_B^{(p)} f$, such that

$$\int_B |f - I_B^{(p)} f|^p \, d\mu = \inf_{c \in \mathbb{R}} \int_B |f - c|^p \, d\mu.$$

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Let $S \subset L^p(X)$, $p > 0$, be bounded set. If for some $q > 0$ the following condition holds

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$$\lim_{r\to 0}\sup_{f\in S}\int_X |I^{(q)}_{B(x,r)}f - f(x)|^p\, d\mu(x) = 0. \qquad (6)$$

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Thank you for attention!