Inequalities in approximation theory involving derivatives of functions in L_p, $0 < p < 1$

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- **The spaces $L_p(\mathbb{T})$:** $f(x+2\pi)=f(x)$, $x\in\mathbb{T}=(-\pi,\pi]$

$$\|f\|_p=\begin{cases}\left(\int\limits_{\mathbb{T}}|f(x)|^p dx\right)^{\frac{1}{p}}, & 0<p<\infty\\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{T}}|f(x)|, & p=\infty\end{cases}$$

- **Modulus of smoothness:**

$$\omega_r(f,\delta)_p=\sup_{|h|\le\delta}\|\Delta_h^r f\|_p$$

where

$$\Delta_h^r f(x)=\Delta_h^1\Delta_h^{r-1}f(x),\quad \Delta_h^1 f(x)=f(x+h)-f(x)$$

- **Error of the best approximation:**

$$E_n(f)_p=\inf_{T_n\in\mathcal{T}_n}\|f-T_n\|_p$$

Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, and $k, r, n \in \mathbb{N}$

- **Direct theorems:**

$$E_{n-1}(f)_p \le C(k,p)\,\omega_k\left(f, \frac{1}{n}\right)_p \qquad (1)$$

$$E_{n-1}(f)_p \le \frac{C(k,r,p)}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \qquad (2)$$

- **Inverse theorems:**

$$\omega_k\left(f, \frac{1}{n}\right)_p \le \frac{C(k,p)}{n^k}\sum_{\nu=0}^{n}(\nu+1)^{k-1}E_\nu(f)_p \qquad (3)$$

$$\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \le C(k,r,p)\left(\frac{1}{n^k}\sum_{\nu=0}^{n}(\nu+1)^{k+r-1}E_\nu(f)_p + \sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)_p\right) \qquad (4)$$

Jackson, Bernstein, Zygmund, Quade, Stechkin, Akhiezer, A.Timan and M.Timan...
(See R.A. DeVore and G.G. Lorentz, Constructive approximation. 1993)

Let $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, and $k, r, n \in \mathbb{N}$

- **Inequalities for the best approximations**

$$E_n(f)_p \le \frac{C(r)}{n^r} E_n\left(f^{(r)}\right)_p \qquad (5)$$

$$E_n\left(f^{(r)}\right)_p \le C(r)\left(n^r E_n(f)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)_p\right) \qquad (6)$$

- **Inequalities for the moduli of smoothness**

$$\omega_{k+r}\left(f, \frac{1}{n}\right)_p \le \frac{1}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \qquad (7)$$

$$\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \le C(k,r)\sum_{\nu=n+1}^{\infty} \nu^{r-1}\omega_{k+r}\left(f, \frac{1}{\nu}\right)_p \qquad (8)$$

Taberskii (1977), Johnen and Scherer (1976)

- **Free knot spline approximation**

- **Def.** Let $\Sigma_{r,n}$ be the set of all piecewise polynomial functions of degree $r-1$ with $n-1$ free knots, i.e. $s \in \Sigma_{r,n}$ if there exist points (knots)

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

and algebraic polynomials $Q_i \in \mathcal{P}_{r-1}$ such that

$$s(x) = Q_i(x) \quad \text{for} \quad x \in (x_{i-1}, x_i), \quad i = 1, \ldots, n$$

- **Theorem** (Petrushev (1988)) Let $0 < q < \infty$, $r \in \mathbb{N}$, $\alpha > 0$, and $\gamma = (\alpha + \frac{1}{q})^{-1}$. Then

$$\inf_{s \in \Sigma_{r,n}} \|f - s\|_{L_q[0,1]} \le \frac{C(\alpha, r, p)}{n^\alpha} \left(\int_0^1 \left(\frac{\omega_r(f,t)_\gamma}{t^\alpha} \right)^\gamma \frac{\mathrm{d}t}{t} \right)^{1/\gamma}$$

- ***Recall.*** Let $f \in L_p$, $1 \le p < \infty$, and $r \in \mathbb{N}$. Then

  (*i*) $\omega_r(f,\delta)_p = o(\delta^r) \quad \Longrightarrow \quad f \equiv \mathrm{const}$

  (*ii*) $\omega_r(f,\delta)_p = \mathcal{O}(\delta^r) \quad \Longleftrightarrow \quad \begin{cases} f \in W_p^r, & p>1 \\ f^{(r-2)} \in AC,\ f^{(r-1)} \in BV, & p=1 \end{cases}$

- **Theorem** (Krotov (1982), Brudnyi (1984))
  Let $f \in L_p$, $0<p<1$, and $r \in \mathbb{N}$. Then

  (*i*) $\omega_r(f,\delta)_p = o(\delta^{r-1+\frac{1}{p}}) \quad \Longrightarrow \quad f \equiv \mathrm{const}$

  (*ii*) $\omega_r(f,\delta)_p = \mathcal{O}(\delta^{r-1+\frac{1}{p}}) \quad \Longleftrightarrow \quad f^{(r-1)}(x) = \sum_{x \le x_k} c_k, \quad \sum_k |c_k|^p < \infty$

- **Theorem** (Krotov (1982))
  Let $0 < p < 1$, $\sigma(\delta) > 0$, and $\lim_{\delta\to 0} \sigma(\delta) = \infty$. Then there exists a continuous nowhere differentiable function f such that

$$\omega_1(f,\delta)_p = \mathcal{O}(\sigma(\delta)\delta^{\frac{1}{p}})$$

- **Theorem** (Storozhenko, Krotov, Oswald (1975))
  If $0 < p < 1$, $f \in AC(\mathbb{T})$, and

$$\omega_1(f,\delta)_p = o(\delta) \quad \text{as} \quad \delta \to 0$$

  then $f \equiv \mathrm{const}$.

Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $k, r, n \in \mathbb{N}$

- **Direct theorems:**

$$E_{n-1}(f)_p \le C(k,p)\,\omega_k\left(f, \frac{1}{n}\right)_p \tag{1}$$

! NOT TRUE:

$$E_{n-1}(f)_p \le \frac{C(k,r,p)}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \tag{2}$$

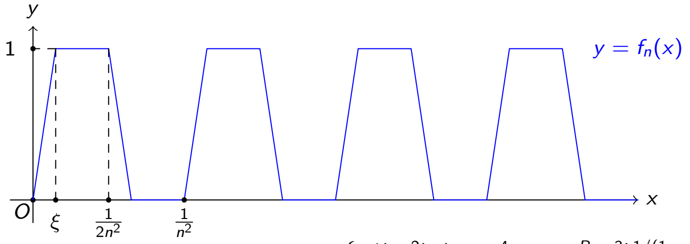
- **Inverse theorems:**

$$\omega_k\left(f, \frac{1}{n}\right)_p \le \frac{C(k,p)}{n^k}\left(\sum_{\nu=0}^{n}(\nu+1)^{kp-1}E_\nu(f)_p^p\right)^{\frac{1}{p}} \tag{3}$$

$$\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \le C(k,r,p)\left(\frac{1}{n^k}\sum_{\nu=0}^{n}(\nu+1)^{(k+r)p-1}E_\nu(f)_p^p + \sum_{\nu=n+1}^{\infty}\nu^{rp-1}E_\nu(f)_p^p\right)^{\frac{1}{p}} \tag{4}$$

Storozhenko, Krotov, and Oswald (1975), Ivanov (1975), Storozhenko and Oswald (1978)

- **Theorem** (Kopotun (1996))
  For every $0 < p < 1$, $A \in \mathbb{R}_+$, $B \in \mathbb{R}$, and $n \in \mathbb{N}$ there exists $f_n \in AC(\mathbb{T})$ such that
  $$E_n(f_n)_p > An^B \|f_n'\|_{L_p(\mathbb{T})}$$



$$0 < \xi \le \{1/(4n^2), (2^{-p-4}A^{-p}n^{-Bp-3})^{1/(1-p)}\}$$

Let $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $k, r, n \in \mathbb{N}$

- **Inequalities for the best approximations**

! NOT TRUE:
$$E_n(f)_p \le \frac{C(r,p)}{n^r} E_n\left(f^{(r)}\right)_p \tag{5}$$

$$E_n\left(f^{(r)}\right)_p \le C(r,p)\left(n^r E_n(f)_p + \left(\sum_{\nu=n+1}^{\infty} \nu^{rp-1} E_\nu(f)_p^p\right)^{\frac{1}{p}}\right) \tag{6}$$

- **Inequalities for the moduli of smoothness**

! NOT TRUE:
$$\omega_{k+r}\left(f, \frac{1}{n}\right)_p \le \frac{C(k,r,p)}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \tag{7}$$

$$\omega_k\left(f^{(r)}, \frac{1}{n}\right)_p \le C(k,r,p)\left(\int_0^{1/n}\left(\frac{\omega_{k+r}(f,t)_p}{t^r}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \tag{8}$$

Ivanov (1975), Petrushev and Popov (1987), Ditzian and Tikhonov (2007)

- **Theorem** (Czipzer and Freud (1958))
  Let $f \in W_p^r$, $1 \le p \le \infty$, $r, n \in \mathbb{N}$, and polynomials $T_n \in \mathcal{T}_n$ be such that $\|f - T_n\|_p = E_n(f)_p$. Then

$$\|f^{(r)} - T_n^{(r)}\|_{L_p(\mathbb{T})} \le C(r) E_n(f^{(r)})_p$$

- **Theorem** (Ditzian (1995))
  For $0 < p < 1$ and $n \in \mathbb{N}$ there exists $f \in AC$ for which we cannot have $T_n \in \mathcal{T}_n$ such that

$$\|f - T_n\|_{L_p(\mathbb{T})} \le C\omega_2\left(f, \frac{1}{n}\right)_p$$

  and

$$\|f' - T_n'\|_{L_p(\mathbb{T})} \le C\omega_1\left(f', \frac{1}{n}\right)_p$$

  simultaneously with a constant C independent of f and n.

- Kopotun (1997): Some positive results for convex functions in the non-periodic case.

- **To summarize**, the following inequalities are not valid if $0 < p < 1$

(A) $$E_{n-1}(f)_p \le \frac{C(k,r,p)}{n^r}\omega_k\left(f^{(r)},\frac{1}{n}\right)_p$$

(B) $$E_n(f)_p \le \frac{C(r,p)}{n^r}E_n\left(f^{(r)}\right)_p$$

(C) $$\omega_{k+r}(f,\delta)_p \le C(k,r,p)\delta^r\omega_k\left(f^{(r)},\delta\right)_p$$

(D) $$\|f^{(r)} - T_n^{(r)}\|_{L_p(\mathbb{T})} \le C(r,p)E_n(f^{(r)})_p$$

- **Main problem:**
  - find analogues of (A)–(D) which can provide

$$\omega_{r+k}(f,\delta)_p = \mathcal{O}(\delta^{r+\alpha}) \iff \omega_k(f^{(r)},\delta)_p = \mathcal{O}(\delta^\alpha)$$

$$E_n(f)_p = \mathcal{O}(n^{-r-\alpha}) \iff E_n(f^{(r)})_p = \mathcal{O}(n^{-\alpha})$$

- **Theorem 1.** Let $0 < p < 1$, $r \in \mathbb{N}$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. Then

$$E_n(f)_p \le \frac{C(r,p)}{n^r} \left(E_n(f^{(r)})_p + \left(\frac{1}{n^{1-p}} \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f^{(r)})_p^p}{\nu^p} \right)^{\frac{1}{p}} \right)$$

- **Corollary 1.** Let $0 < p < 1$, $r \in \mathbb{N}$, $\alpha > \frac{1}{p} - 1$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. Then the following assertions are equivalent:

$$(i) \qquad E_n(f)_p = \mathcal{O}(n^{-r-\alpha}), \quad n \to \infty$$

$$(ii) \qquad E_n(f^{(r)})_p = \mathcal{O}(n^{-\alpha}), \quad n \to \infty$$

- **Theorem 2.** Let $0 < p < 1$, $r, k, m \in \mathbb{N}$, and $f^{(r-1)} \in AC(\mathbb{T})$. Then

$$\omega_{r+k}(f, \delta)_p \le C(r,k,m,p)\delta^r \left(\omega_k(f^{(r)}, \delta)_p + \left(\delta^{1-p} \int_0^\delta \frac{\omega_m(f^{(r)}, t)_p^p}{t^{2-p}} dt \right)^{\frac{1}{p}} \right)$$

- **Corollary 2.** Let $0 < p < 1$, $r \in \mathbb{N}$, $\alpha > \frac{1}{p} - 1$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. Then the following assertions are equivalent:

$$(i) \qquad \omega_{r+k}(f, \delta)_p = \mathcal{O}(\delta^{r+\alpha}), \quad \delta \to 0$$

$$(ii) \qquad \omega_k(f^{(r)}, \delta)_p = \mathcal{O}(\delta^{\alpha}), \quad \delta \to 0$$

- **Proposition 1.** Let $0 < p < 1$, $r \in \mathbb{N}$, and $\gamma > 0$. Then for any $A \in \mathbb{R}$, $C > 0$, and $n \geq n_0$ there exists a function $f_0 \in C^{r-1}(\mathbb{T})$ such that

$$E_n(f_0)_p > Cn^A \left(E_n(f_0^{(r)})_p^p + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f_0^{(r)})_p^p}{\nu^{p+\gamma}} \right)^{\frac{1}{p}}$$

and

$$\omega_{r+k}(f_0,\delta)_p > C\delta^A \left(\omega_k(f_0^{(r)},\delta)_p^p + \int_0^\delta \frac{\omega_m(f_0^{(r)},t)_p^p}{t^{2-p-\gamma}}\,dt \right)^{\frac{1}{p}}$$

- **Corollary 3. (The second Jackson theorem for $0 < p < 1$)**
  Let $0 < p < 1$, $r, k \in \mathbb{N}$, and $f^{(r-1)} \in AC(\mathbb{T})$. Then

$$E_n(f)_p \le \frac{C(p,k,r)}{n^{r+\frac{1}{p}-1}} \left(\int_0^{1/n} \frac{\omega_k(f^{(r)},t)_p^p}{t^{2-p}} dt \right)^{\frac{1}{p}}$$

- **Corollary 4.** Let $0 < p < 1$, $k, r \in \mathbb{N}$, $\frac{1}{p} - 1 < \alpha < k$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. Then the following assertions are equivalent:

  (i) $\quad E_n(f)_p = \mathcal{O}(n^{-r-\alpha}), \quad n \to \infty$

  (ii) $\quad \omega_k(f^{(r)}, \delta)_p = \mathcal{O}(\delta^{\alpha}), \quad \delta \to 0$

- **Theorem 3.** Let $0 < p < 1$, $r \in \mathbb{N}$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. If $T_n \in \mathcal{T}_n$ is such that $\|f - T_n\|_p = E_n(f)_p$, then

$$\|f^{(r)} - T_n^{(r)}\|_p \le C(r,p)\left(E_n(f^{(r)})_p + \left(\frac{1}{n^{1-p}}\sum_{\nu=n+1}^{\infty}\frac{E_\nu(f^{(r)})_p^p}{\nu^p}\right)^{\frac{1}{p}}\right)$$

- **Corollary 5.** Let $0 < p < 1$, $r \in \mathbb{N}$, $\alpha > \frac{1}{p} - 1$, and let f be such that $f^{(r-1)} \in AC(\mathbb{T})$. Then the following assertions are equivalent:

$$(i) \qquad E_n(f)_p = \mathcal{O}(n^{-r-\alpha}), \quad n \to \infty$$

$$(ii) \qquad E_n(f^{(r)})_p = \mathcal{O}(n^{-\alpha}), \quad n \to \infty$$

$$(iii) \qquad \|f^{(r)} - T_n^{(r)}\|_p = \mathcal{O}(n^{-\alpha}), \quad n \to \infty$$

Thank you for attention!