Calderón-Zygmund operators in new function spaces "near L^1". Applications to the BVPs

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1. Preliminaries

1.1 The class of exponents $\mathbb{P}(\Gamma)$

Let Γ be a simple closed rectifiable curve. We say that a given on Γ function $p = p(t)$ belongs to the class $\mathbb{P}(\Gamma)$ if it satisfies the log-continuity condition i. e.

(i) there exists a number B such that for any t_1 and t_2 from Γ we have

$$|p(t_1) - p(t_2)| < \frac{B}{|\ln|t_1 - t_2||}; \tag{1}$$

and

(ii)

$$p_- = \min_{t\in\Gamma} p(t) > 1. \tag{2}$$

By $\mathbb{P}_1$ we denote a set of exponents $p(t)$ for which the condition $\min_{t\in\Gamma} p(t) = 1$ together with (1) holds.

1.2 Variable exponent Lebesgue spaces

By $L^{p(t)}(\Gamma)$ we denote the Banach function space of measurable on Γ function f such that $\|f\|_{p(\cdot)} < \infty$, where

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_0^l \left|\frac{f(t(s))}{\lambda}\right|^{p(t(s))} ds \leq 1\right\}.$$

Here, $t = t(s)$, $0 \leq s \leq l$, is the equation of the curve Γ with respect to the arc abscissa s.

In recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It has become necessary to introduce and study the news spaces from different viewpoints. One of such spaces is the variable exponent Lebesgue spaces. The extensive investigation of these spaces widely stimulated by appeared applications in various problems of applied mathematics, e.g. in modelling of elctrorheological fluids and more recently in image restoration.

2. An invariant class with respect to the Cauchy singular integral operator

To discuss the BVP in the framework of variable exponent Lebesgue spaces $L^{p(t)}$, $p \in \mathbb{P}_1(\Gamma)$ we need to explore some invariant subclasses of $L^{p(t)}$ with respect to the following singular operator

$$\mathcal{S}_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t).$$

Note that in the case of smooth curves this singular integral coincides a. e. with the Cauchy singular integral $S_\Gamma f(t)$.

Throughout this section we assume that Γ is a simple closed Carleson (regular) curve.

Let $d = \mathrm{diam}\Gamma$, $D(t, r) := B(t, r) \cap \Gamma$, $t \in \Gamma$, $0 < r < \mathrm{diam}\Gamma$, $B(t, r)$ be a ball with center $t \in \Gamma$ and radius r.

Let us introduce the following characteristics for functions $f \in L^{p(\cdot)}(\Gamma)$, $p \in \mathbb{P}_1(\Gamma)$:

$$\Omega(f,\rho)_{L^{p(\cdot)}(\Gamma)} = \sup_{0<r\leq\rho} \left\| \frac{1}{r} \int\limits_{D(t,r)} f(\tau)d\tau - f(t) \right\|_{L^{p(\cdot)}(\Gamma)}$$

and

$$\Omega^*(f,\delta) = \sup_{\rho\geq\delta} \frac{\Omega(f,\rho)}{\rho}.$$

The following statements hold:

Theorem 1. Let $p \in \mathbb{P}_1$, i. e. $\min p(t) = 1$. Let for $f \in L^{p(\cdot)}(\Gamma)$ the condition

$$\int\limits_0^d \frac{\Omega^*(f,\delta)_{L^{p(\cdot)}(\Gamma)}}{\delta} d\delta < +\infty \quad (V_0 \text{ condition})$$

be satisfied. Then $\mathcal{S}_\Gamma f \in L^{p(\cdot)}(\Gamma)$.

Theorem 2. Let $p \in \mathbb{P}_1$. For $f \in L^{p(\cdot)}(\Gamma)$ the following Zygmund type inequality

$$\Omega^*(\mathcal{S}_\Gamma f, \delta) \leq c \left(\int_0^\delta \frac{\Omega^*(f,h)}{h} dh + \delta^2 \int_\delta^d \frac{\Omega^*(f,h)}{h^3} dh \right)$$

holds. Let us introduce the following classes of functions:

$$V_k := \left\{ f \in L^{p(\cdot)}(\Gamma) : \int_0^d \frac{\Omega^*(f,\delta)_{L^{p(\cdot)}(\Gamma)}}{\delta} \left(\ln \frac{d}{\delta} \right)^k d\delta < +\infty \right\}, \ k = 0, 1 \ldots$$

Corollary of Theorem 2. *Let $p \in \mathbb{P}_1$, $p \in \mathbb{P}_1(\Gamma)$ and $f \in V_k$ ($k \in \mathbb{N}$). Then $\mathcal{S}_\Gamma f \in V_{k-1}$.*

Define the following set $\quad V = \bigcap_{k=0}^\infty V_k$.

Corollary. Let $p \in \mathbb{P}_1$. Then the class V is invariant towards the singular integral i. e.

$$f \in V \Longrightarrow \mathcal{S}_\Gamma f \in V.$$

3. The Riemann BVP in the framework of variable exponent Lebesgue spaces $L^{p(t)}$ when $p \in \mathbb{P}_1(\Gamma)$ i. e. $\min p(t) = 1$

Now our aim is to apply the latter results to two types boundary value problems to analytic functions:
i) The Riemann problem: find the function Φ from a given class of analytic functions on the plane, cut along Γ – a rectifiable curve, which boundary values satisfy the condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{3}$$

where G and g are functions prescribed on Γ, and Φ^+ and Φ^- are boundary values of Φ on Γ.
ii) The Riemann-Hilbert problem (one-sided problem)

$$\mathrm{Re}[a(t)\Phi^+(t)] = b(t), \quad t \in \Gamma, \tag{4}$$

In the sequel we will employ the following classes of analytic functions

$$K^{p(t)}(\Gamma) := \{\Phi : \Phi(z) = K_\Gamma(\varphi)(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - z} d\tau + q(z),$$

$\varphi \in L^{p(t)}(\Gamma)\}$, $p \in \mathbb{P}_1(\Gamma)$, $q(z)$ is a polynomial and

$$K^{p(t)}(\Gamma; V) = \{\Phi : \Phi(z) = K_\Gamma(\varphi)(z), \quad \varphi \in V(\Gamma)\}$$

By definition a continuous on Γ function ψ belongs to the generalized Dini class $D^*(\Gamma)$ class if

$$\int\limits_0^d \frac{\omega^*(\psi,\delta)}{\delta} d\delta < \infty, \text{where } \omega^*(\psi,\delta) := \sup_{h \geq \delta} \frac{\omega(\psi,h)}{h},$$

here $\omega(\psi, h) = \sup\limits_{|t-\tau| \leq h} |\psi(t) - \psi(\tau)|.$

The existence of solutions for the Riemann and Riemann-Hilbert problems is known in fairly general setting of arbitrary Jourdan domains, measurable coefficients and measurable data. The obtained result is formulated in terms of harmonic measure and principal asymptotic values.The space of solutions is of infinite dimension (see V. Ryazanov 2014).
In our talk our goal is to focus the study of constructional and quantitive aspects and to solve the the considered problems in the spirit of the concepts developed by F. Gakhov, N. Muskhelishvili and I. Vekua.

We emphasize that the Riemann and Riemann-Hilbert problems with data from $L^{p(t)}$. $\min p(t) > 1$ are solved in speaker's joint with V.Paatashvili papers. We refer to the papers published in: Boundary Value Problems (2005) Complex Analysis and Operator

Theory (2008)

Georgian Math.J. (2009) Complex Variables and Elliptic Equations

(2017) (accepted)

Math.Methods in Appl.Sciences (2017) (accepted). In

above-mentioned papers are done the complete pictures of solutions of the problems under different assumptions on the coefficient $G(t)$ (including strongly oscillating case) in the domains with non-smooth boundaries (in some cases including cusps): the necessary and sufficient solvability condition is established, an influence of boundary's geometry on solvability picture are revealed and the solutions in explicit form are constructed when they exist.

Bellow we give an example of the wide class of oscillating coefficients $G(t)$, for which the above-mentioned results has been realized.
Let $p \in \mathbb{P}(\Gamma)$, i. e. $\min p(t) > 1$. We say that $G \in A(p(t), \Gamma)$ if for every point $\tau \in \Gamma$, there exists an arc $\Gamma_\tau \subset \Gamma$ containing the point τ at which almost all values of the function G lie inside the angle with vertex at the origin of coordinates of the size

$$\alpha_\tau = 2\pi \left[\sup_{t \in \Gamma_\tau} \max \left(p(t), p'(t) \right)^{-1} \right]$$

This class of coefficients is an extension of well-known Simonenko's class and it is well suited to the variable exponent case.

Let us return to the Riemann problem in the framework of $K^{p(t)}(\Gamma)$ and $K^{p(t)}(\Gamma; V)$ classes.

It should be noted that the Riemann problem is unsolvable in general in $K^{p(\cdot)}(\Gamma)$, $p \in \mathbb{P}_1$ for arbitrary $g \in L^{p(\cdot)}(\Gamma)$, even in the case $g \in D^*(\Gamma)$ and $\operatorname{ind} G = 0$.

We consider the Riemann boundary value problem for analytic function into the following two settings:
i) A sought function $\Phi(z)$ belongs to $K^{p(t)}(\Gamma)$, $g(t) \in V_0(\Gamma)$.
ii) A sought function Φ is taken from $K^{p(t)}(\Gamma, V(\Gamma))$ and $g \in V(\Gamma)$.

The following theorem is true:

Theorem 3. Let Γ be a Carleson curve, $p(\cdot) \in \mathbb{P}_1(\Gamma)$, $G(t) \neq 0$, $t \in \Gamma$, $G \in D^*(\Gamma)$. Then

i) If ind $G(t) = \varkappa = 0$, $g \in V_0(\Gamma)$ ($g \in V(\Gamma)$). Then the problem (3) is uniquely solvable in $K^{p(\cdot)}(\Gamma)$ ($K^{p(\cdot)}(\Gamma, V(\Gamma))$ and the solution is given by the formula

$$\Phi(z) = X(z) \int_{\Gamma} \frac{g(t)}{X^{+}(t)} \frac{dt}{t-z}, \tag{5}$$

where

$$X(z) = \begin{cases} \exp h(z), & z \in D; \\ (z-z_0)^{-\varkappa} \exp h(z), & z \in D^{-}, z_0 \in D^{+}. \end{cases}$$

and

$$h(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln G(t)(t-z_0)^{-\varkappa}}{t-z} dt;$$

ii) When ind $G(t) = \varkappa > 0$ the problem (3) is non-uniquely solvable and all solutions are given by the formula

$$\Phi(z) = X(z) \int\limits_{\Gamma} \frac{g(t)}{X^{+}(t)} \frac{dt}{t-z} + X(z) Q_{\varkappa-1}(z), \qquad (6)$$

where $Q_{\varkappa-1}(z)$ is an arbitrary polynomial of degree $\varkappa - 1$ $(Q_{-1} \equiv 0)$;
iii) When ind $G(t) = \varkappa < 0$ the problem (3) is solvable if and only if

$$\int\limits_{\Gamma} \frac{g(t)}{X^{+}(t)} t^{k} dt = 0, \quad k = \overline{0, \varkappa - 1}; \qquad (7)$$

Under these conditions problem (3) is uniquely solvable and the solution is given by the formula (5).

4. The Riemann-Hilbert problem in the case $p \in \mathbb{P}_1(\Gamma)$

In this section we assume that a simply connected domain D is bounded by the curve Γ satisfying the Dini condition

$$\int_0^{\delta_0} \frac{\omega(\theta,\delta)}{\delta} d\delta < +\infty,$$

where $\omega(\theta,\delta)$ is the moduli of continuity of the angle $\theta(t)$ which is the angle between the tangent at point t, $t \in \Gamma$ and real axis.
It is well-known that in this case the conformal mapping satisfies the condition

$$0 < m \leq |\psi'(w)| \leq M < \infty,$$

where m and M don't depend on w, $|w| < 1$.
Let us consider the Riemann-Hilbert problem

$$\mathrm{Re}[a(t)\Phi^+(t)] = b(t), \quad t \in \Gamma, \tag{8}$$

where $a(t)$ and $b(t)$ are measurable functions on Γ.

The problem (8) is explored in two different settings.

Theorem 4. Let $p(t) \in \mathbb{P}_1(\Gamma)$ and

i) $G(t) \neq 0$, $G^*(t) \in D(\Gamma)$, where $G(t) = \frac{\overline{a(t)}}{a(t)}$; $g(t) \in V_0(\Gamma)$, $g(t) = \frac{2b(t)}{a(t)}$;

ii) $G(t) \in D^*(\Gamma)$, $g(t) \in V(\Gamma)$.

Then:

when $\varkappa \geq 0$, then problem (8) is solvable under the conditions *i*) in $K^{p(\cdot)}$ and when conditions *ii*) are satisfied then it is solvable in $K^{p(t)}(\Gamma; V)$ class and in both cases the solutions are given by the formula

$$\Phi(z) = \frac{1}{2}\left[K_\Gamma\left(\frac{2b}{aX^+}\right)(z) + \left(\overline{K_\Gamma \frac{2b}{aX^+}}\right)\left(\frac{1}{\overline{z}}\right)\right] + X(z)\left(c_0 + c_1 z + \cdots + c_\varkappa z^\varkappa\right),\ z \in D,$$

where $X(z) = \begin{cases} \exp h(z), & z \in V; \\ z^{-\varkappa}\exp h(z), & |z| > 1, \end{cases}$,

$h(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\ln G(t)(t-z_0)^{-\varkappa}}{t-z}dt$ and $c_0, c_1, \ldots, c_\varkappa$ are arbitrary numbers such that $c_k = \overline{c}_{\varkappa-k}$, $\quad k = \overline{0, \varkappa}$.

When $\varkappa < 0$ problem (8) is solvable if and only if

$$\int\limits_{\Gamma} \frac{2b(t)}{a(t)X^{+}(t)} t^k dt = 0, \quad k = \overline{0, |\varkappa|}$$

and the unique solution is given by the formula

$$\Phi(z) = \frac{1}{2\pi i} X(z) \int\limits_{\Gamma} \frac{2b(t)}{a(t)X^{+}(t)} \frac{dt}{t - z}.$$

5. An extension of $L^{p(t)}$ integrability. Application to the Riemann problem

In the sequel we use the following notation

$$\mathcal{L}^{p(t)}(\Gamma) := L^{p(t)}(\Gamma) \quad \text{when} \quad \min_{t\in\Gamma} p(t) = 1.$$

Cauchy singular integral operator S_Γ is bounded in $L^{p(t)}$ if and only if $\min\limits_{t\in\Gamma} p(t) > 1$. Thus $\mathcal{L}^{p(t)}$ is non-invariant with respect to the operator S_Γ.

Definition. A measurable function f on Γ we call as $\widetilde{\mathcal{L}}^{p(t)}$-integrable if f can be represented as

$$f = h + S_\Gamma g, \quad h \in \mathcal{L}^{p(t)}(\Gamma), \quad g \in \mathcal{L}^{p(t)}(\Gamma)$$

and we set

$$\left(\widetilde{\mathcal{L}}^{p(t)}\right) \int\limits_\Gamma f(t)dt := \int\limits_\Gamma (h(t))^{p(t)} dt.$$

It is evident that every $f \in \mathcal{L}^{p(t)}$ is $\widetilde{\mathcal{L}}^{p(t)}$-integrable. Moreover, for arbitrary $f \in \mathcal{L}^{p(t)}$ we have $S_\Gamma f \in \widetilde{\mathcal{L}}^{p(t)}$. Therefore, $\widetilde{\mathcal{L}}^{p(t)}$ is invariant with respect to the operator S_Γ.

Theorem 5. Let D^+ be a bounded domains with piecewise smooth boundary Γ. The Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(t)dt}{t-z}, \quad \varphi \in \mathcal{L}^{p(t)}(\Gamma), \quad z \in D^+,$$

is $\widetilde{\mathcal{L}}^{p(t)}$ -Cauchy integral i. e.

$$\Phi(z) = \frac{1}{2\pi i} \left(\widetilde{\mathcal{L}}^{p(t)}\right) \int\limits_{\Gamma} \frac{\Phi^+(t)dt}{t-z}$$

Theorem 6. A class of functions representable in $D^+ \bigcup D^-$ by the $\widetilde{\mathcal{L}}^{p(t)}$-Cauchy type integrals coincides with the class of functions representable in the form

$$F(z) = \begin{cases} F_1(z), & \text{for} \quad z \in D^+ \\ F_2(z), & \text{for} \quad z \in D^-, \end{cases}$$

where $F_i(z) = K_\Gamma(f_i)(z)$, $f_i \in L(\Gamma)$, $i = 1, 2$.

As it was already mentioned the Riemann problem

$$\Phi^+(t) + G(t)\Phi^-(t) = g(t),$$

$G \in C(\Gamma)$, $G \neq 0$ and $g \in \mathcal{L}^{p(t)}(\Gamma)$ is not solvable for arbitrary $g \in \mathcal{L}^{p(t)}(\Gamma)$ in the class of Cauchy type integrals with densities from $\mathcal{L}^{p(t)}(\Gamma)$. This fact caused by the non-invariance of $\mathcal{L}^{p(t)}$ with respect to the operator S_Γ.

The idea of $\widetilde{\mathcal{L}}^{p(t)}$ integrability and the properties of $\widetilde{\mathcal{L}}^{p(t)}$- Cauchy type integrals allow us to state

Theorem 7. The Riemann problem is solvable for arbitrary $g \in \mathcal{L}^{p(t)}$ in the class of $\widetilde{\mathcal{L}}^{p(t)}$-Cauchy type integrals with polynomial principal part at infinity and all solutions are expressed by the formula

$$\Phi(z) = \frac{X(z)}{2\pi i}\left(\widetilde{\mathcal{L}}^{p(t)}\right)\int_\Gamma \frac{[X(\tau)]^{-1}g(\tau)d\tau}{\tau - z} + X(z)q(z)$$

with $X(z)$ defined by the formula from previous section and $q(z)$-is some polynomial.