Orthonormal spline systems on $\mathbb{R}$ with zero means as basis in $H^1(\mathbb{R})$

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New perspectives in the theory of function spaces and their applications

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Let
$\mathcal{T} = \{0, 1,\ 1/2,\ 1/4, 3/4,\ 1/8, 3/8, 5/8, 7/8, ...\} = \{d_n\}_{n=0}^{\infty}$
be the sequence of dyadic grid points.
Denote by $\mathcal{T}_n = \{d_i : 0 \le i \le n\}$.

- $0 = d_{n,1} < d_{n,2} < \ldots < d_{n,n+1} = 1,$
- $\{d_{n,i} : 1 \le i \le n+1\} = \mathcal{T}_n.$

Definition of the classical Franklin system

Denote by S_n the space of all continuous functions on $[0,1]$, which are linear on each $[d_{n,i}; d_{n,i+1}]$, $i = 1, 2, ..., n$.

Clearly, $S_{n-1} \subset S_n$ and S_{n-1} has codimension 1 in S_n.

There exists a unique function up to sign $f_n \in S_n$ such that

$$f_n \perp S_{n-1}, \quad \|f_n\|_2 = 1.$$

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Definition

The system of functions $(f_n)_{n=0}^{\infty}$ *is called classical Franklin system .*

Results for classical Franklin system

1. In 1975 S. Bochkarev proved that classical Franklin system is an unconditional basis in $L^p[0,1],\ 1 < p < \infty$.
2. In 1982 P. Wojtaszczyk showed that classical Franklin system is an unconditional basis in $H^1[0,1]$.

Definition of general Franklin systems

Definition

Let $\mathcal{T} = \{t_n : n \geq 0\}$ *be a sequence of points from* $[0,1]$. *The sequence* $\mathcal{T}$ *is called* **admissible**, *if*

- $t_0 = 0, t_1 = 1, t_n \in (0,1)$
- $t_i \neq t_j$, *if* $i \neq j$,
- $\mathcal{T}$ *is dense in* $[0,1]$.

For an admissible sequence of points $\mathcal{T} = (t_n : n \geq 0)$ and $n \geq 1$ denote $\mathcal{T}_n = \{t_i : 0 \leq i \leq n\}$.

- $0 = \tau_{n,1} < \tau_{n,2} < \ldots < \tau_{n,n+1} = 1$,
- $\{\tau_{n,i} : 1 \leq i \leq n+1\} = \mathcal{T}_n$.

Definition of general Franklin systems

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Definition

The system of functions $(f_n)_{n=0}^{\infty}$ is called general Franklin system corresponding to the partition $\mathcal{T}$.

Results for general Franklin systems

1. The unconditionality of general Franklin system in $L^p[0,1],\ 1 < p < \infty$ for any admissible sequence was proved by G. Gevorkyan and A. Kamont in 2004.
2. G. Gevorkyan and A. Kamont found characterizations of all those sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$ in 2005.

Regularity Conditions

Definition

Let $\mathcal{T}$ be an admissible sequence of knots. We say that $\mathcal{T}$ is r-**regular** *with parameter $\gamma \geq 1$, if for any $n \in \mathbb{N}$*

$$\gamma^{-1} \leq \frac{\nu_{n,i+1}}{\nu_{n,i}} \leq \gamma, \quad \text{for} \quad i = 1, ..., n,$$

where $\nu_{n,i} = \tau_{n,i+r} - \tau_{n,i} = \big|[\tau_{n,i}, \tau_{n,i+r}]\big|$.

Franklin system as a basis and unconditional basis in $H^1[0,1]$.

Theorem (G. Gevorkyan, A. Kamont, 2005)

Franklin system is a basis in $H^1[0,1]$ iff $\mathcal{T}$ is 2-regular.

Theorem (G. Gevorkyan, A. Kamont, 2005)

Franklin system is an unconditional basis in $H^1[0,1]$ iff $\mathcal{T}$ is 1-regular.

Definition of orthonormal spline system with zero mean

Denote by S_n^r the space of all $r-2$ times continuously differentiable functions on $[0,1]$, which are polynomials of degree less than r on each $[\tau_{n,i};\tau_{n,i+1}]$.

Clearly, $S_{n-1}^r \subset S_n^r$ and S_{n-1}^r has codimension 1 in S_n^r.

There exists a unique function up to sign $f_n^{(r)} \in S_n^r$ such that

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Definition

The system of functions $(f_n^{(r)})_n$ is called orthonormal spline system of order r corresponding to the sequence $\mathcal{T}$.

Orthonormal spline system as a basis and unconditional basis in $L^p[0,1]$, $H^1[0,1]$.

Theorem (M. Passenbrunner, 2014)

Orthonormal spline system is a basis in $L^p[0,1]$ for any admissible sequence $\mathcal{T}$.

Theorem (G. Gevorkyan, A. Kamont, 2008)

Orthonormal spline system is a basis in $H^1[0,1]$ iff $\mathcal{T}$ is r-regular.

Theorem (G. Gevorkyan, A. Kamont, K.K., M. Passenbrunner, 2015)

Orthonormal spline system is an unconditional basis in $H^1[0,1]$ iff $\mathcal{T}$ is $(r-1)$-regular.

Definition of orthonormal spline system with zero mean

Definition

Let $\mathcal{T} = \{t_n : n \geq 0\}$ be a sequence of points from $\mathbb{R}$. The sequence $\mathcal{T}$ is called **admissible**, *if*

- $t_i \neq t_j$, *if* $i \neq j$,
- $\mathcal{T}$ *is dense in* $\mathbb{R}$.

Let $r \in \mathbb{N}$. For an admissible sequence of points $\mathcal{T} = (t_n : n \geq 0)$ and $n \geq 1$ denote $\mathcal{T}_n = \{t_i : 1 \leq i \leq n + r + 1\}$.

- $\tau_{n,1} < \tau_{n,2} < \ldots < \tau_{n,n+r+1}$,
- $\{\tau_{n,i} : 1 \leq i \leq n + r + 1\} = \mathcal{T}_n$.

Definition of orthonormal spline system with zero mean

Denote by $S_n^{r,0}$ the space of all $r-2$ times continuously differentiable functions on $\mathbb{R}$ **with zero mean** and with supports in $\Delta_n = [\tau_{n,1}; \tau_{n,n+r+1}]$ which are polynomials of degree less than r on each $[\tau_{n,i}; \tau_{n,i+1}]$, $i = 1, 2, ..., n+r$.

Clearly, $S_{n-1}^{r,0} \subset S_n^{r,0}$ and $S_{n-1}^{r,0}$ has codimension 1 in $S_n^{r,0}$.

There exists a unique function up to sign $F_n^{(r)} \in S_n^{r,0}$ such that

$$F_n^{(r)} \perp S_{n-1}^{r,0}, \quad \|F_n^{(r)}\|_2 = 1.$$

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Definition

The system of functions $(F_n^{(r)})_{n=1}^{\infty}$ is called orthonormal spline system with zero means on $\mathbb{R}$ of order r corresponding to the sequence $\mathcal{T}$.

Regularity Conditions

Definition

Let $\mathcal{T}$ be an admissible sequence of knots. We say that $\mathcal{T}$ is r**-regular on** $\mathbb{R}$ *with parameter* $\gamma \geq 1$*, if*

- $\mathcal{T}$ *is* r*-regular,* $\gamma^{-1} \leq \dfrac{\nu_{n,i+1}}{\nu_{n,i}} \leq \gamma$,
- $\dfrac{\nu_{n,1}}{|\Delta_n|} \geq \gamma^{-1}, \quad \dfrac{\nu_{n,n+1}}{|\Delta_n|} \geq \gamma^{-1}$ *for any* $n \in \mathbb{N}$,

where $\nu_{n,i} = \tau_{n,i+r} - \tau_{n,i} = \left|[\tau_{n,i}, \tau_{n,i+r}]\right|$ *and* $\Delta_n = [\tau_{n,1}; \tau_{n,n+r+1}]$.

Franklin system with zero mean as a basis and unconditional basis in $H^1(\mathbb{R})$.

The special case of orthonormal spline system with zero means is Franklin system with zero means corresponding to $r = 2$.

Theorem (G. Gevorkyan, K. K., 2016)

Franklin system with zero mean is a basis in $H^1(\mathbb{R})$ iff $\mathcal{T}$ is 2-regular on $\mathbb{R}$.

Theorem (K. K., 2017)

Franklin system with zero mean is an unconditional basis in $H^1(\mathbb{R})$ iff $\mathcal{T}$ is 1-regular on $\mathbb{R}$.

Orthonormal spline system with zero mean as a basis in $H^1(\mathbb{R})$.

Theorem (K. K., 2017)

Orthonormal spline system with zero mean is a basis in $H^1(\mathbb{R})$ *iff* $\mathcal{T}$ *is* r*-regular on* $\mathbb{R}$.

Question

Is an orthonormal spline system with zero mean an unconditional basis in $H^1(\mathbb{R})$ iff $\mathcal{T}$ is $(r-1)$-regular on $\mathbb{R}$?

Idea of proof. Case $r = 2$

Decomposition

$F_n = f_{n,1} + f_{n,2}$, with

$$|f_{n,1}(x)| \lesssim_\gamma \frac{|J_n|^{1/2}}{|\Delta_n|} q^{e_n(J_n)}, \tag{1}$$

$$|f_{n,1}(x) - f_{n,1}(y)| \lesssim_\gamma \frac{|J_n|^{1/2}}{|\Delta_n|} q^{e_n([x,y]) + e_n(J_n)}, \tag{2}$$

$$|f_{n,2}(x)| \lesssim \frac{|J_n|^{1/2}}{|J_n| + \mathrm{dist}(x, J_n)} \cdot q^{d_n(x)}, \tag{3}$$

where J_n is the characteristic (grid point) interval, $q \in (0,1)$
$d_n(x)$ number of grid points between J_n and x, and
$e_n(I)$ is the minimum of the number of grid points between endpoints of interval I and endpoints of Δ_n

Smoothness of Kernels

Let $\mathcal{T}$ be a 1-regular sequence on $\mathbb{R}$ with parameter γ, and $K_\omega(x,y) := \sum_{n=1}^{\infty} \omega_n F_n(x) F_n(y)$, for $\omega \in \{-1,1\}^N$. There exist constants $\varepsilon > 0$ and C_γ such that

$$|K_\omega(x,y) - K_\omega(x',y)| \leq C_\gamma \frac{|x-x'|^\varepsilon}{|x-y|^{1+\varepsilon}},$$

if $(2\gamma^2+1)|x-x'| < |x-y|$.

Thank you for your attention!