Poincaré inequalities and compact embeddings from Newtonian Sobolev spaces into weighted L^q -spaces

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New perspectives in the theory of function spaces and their applications, Będlewo (Poland), September 17-23, 2017

General assumptions (G):

7. $1 \leq p < \infty$ and $X = (X, d, \mu)$ - a metric space equipped with a metric d and measure μ;
8. μ(*domain measure*) is positive, complete, Borel, locally finite;
9. ν(*target measure*)- is positive, locally finite, complete, Borel measure on X;
10. $E \subset X$- bounded, nonempty, ν-measurable;
11. $1 \leq p, q < \infty$ are given.

The p-weak upper gradients

Definition (Koskela,MacManus, 1998, extensions by Shanmugalingam, 2001)

Let g be a nonnegative Borel function on X

1. *g is an upper gradient of an extended real-valued function f on X if for all rectifiable curves $\gamma : [0, l_\gamma] \to X$,*

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds, \tag{1}$$

where ds is the arc length parametrization of γ.

2. *g is a p-weak upper gradient of f if (1) holds for p-almost every curve, i.e. there exists $\rho \in L^p(X, \mu)$ such that $\int_\gamma \rho\, ds = \infty$ for all curves γ where (1) fails.*

Known:

1. If f has an upper gradient in $L^p_{loc}(X,\mu)$, then it has an a.e. unique *minimal p-weak upper gradient* $g_f \in L^p_{loc}(X,\mu)$ in the sense that $g_f \leq g$ μ-a.e. for every p-weak upper gradient $g \in L^p_{loc}(X,\mu)$ of f (Shanmugalingam);
2. Every p-weak upper gradient g can be approximated by upper gradients g_j so that $\|g_j - g\|_{L^p(X,\mu)} \to 0$ (Koskela, MacManus).

Newtonian space, Dirichlet space

Definition

1. *Whenever* $u \in L^p(X,\mu)$*, let*

$$\|u\|_{N^{1,p}(X,\mu)} = \left(\int_X |u|^p \, d\mu + \inf \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all (p-weak) upper gradients g *of* u*. The Newtonian space on* X *is the space*

$$N^{1,p}(X,\mu) = \{u : \|u\|_{N^{1,p}(X,\mu)} < \infty\}.$$

2. *The Dirichlet space* $D^p(X,\mu)$ *as the set of all* μ*-measureable functions on* X *having an upper gradient in* $L^p(X,\mu)$*, and equip it with the seminorm* $\|g_u\|_{L^p(X,\mu)}$*.*

Known:

1. The quotient space $N^{1,p}(X,\mu)|_{\sim}$, where $u \sim v$ if $\|u-v\|_{N^{1,p}(X,\mu)} = 0$, is a Banach space.
2. If $\Omega \subset \mathbf{R}^n$ is open then $N^{1,p}(\Omega, dx) = W^{1,p}(\Omega)$ (classical Sobolev space).
3. If $d\mu = w\,dx$, where $w > 0$ is locally integrable and $w^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega)$, i.e. w belongs to the $B_p(\Omega)$ class, introduced by Kufner and Opic in 1984, then

$$N^{1,p}(\Omega,\mu) = W^{1,p}(\Omega,\mu)$$

(classical weighted Sobolev space) and $g_u = |\nabla u| \quad$ a.e. (Heinonen–Kilpeläinen–Martio book).

Definition

1. *A measure* μ *is doubling if there exists a constant* $C_\mu \geq 1$, *such that* $\mu(2B) \leq C_\mu \mu(B)$ *holds for all balls* B *in* X.
2. μ *supports a p-Poincaré inequality if there exist constants* $C > 0$ *and* $\lambda \geq 1$ *such that for all balls* $B \subset X$, *all integrable functions* u *on* X *and for all (p-weak) upper gradients* g *of* u,

$$\fint_B |u - u_{B,\mu}| \, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}. \qquad (2)$$

Compactness of embedding. Set of assumptions **(A)**

Given set $\mathcal{Y} \subset D^p_{\rm loc}(X,\mu)$ we consider the following assumptions, in addition to the General assumptions **(G)**:

A. *Existence of a countable sequence of covering families for bounded set E with a controlled overlap and a subordinate Poincaré type inequality valid for $\mathcal{Y}$*:

- E is bounded and there exist positive numbers $r_m \to 0$, as $m \to \infty$, such that for every $r := r_m \leq r_0$, there exists a finite *covering r-family* $\{E_i, E_i'\}_{i=1}^{K(r)}$ consisting of the collections

$\{E_i\}_{i=1}^{K(r)}$ of ν-measurable sets covering E up to a set of ν-measure zero;

$\{E_i'\}_{i=1}^{K(r)}$ of μ-measurable sets such that $E_i \subset E_i'$, possibly depending on r.

1. Poincaré inequality holds for every $r = r_m$- fixed:

$$\left(\int_{E_i} |u - a_{E_i}(u)|^q d\nu\right)^{1/q} \leq C(r)\left(\int_{E_i'} g_u^p d\mu\right)^{1/p}$$

for some 'functional' $a_{E_i} : \mathcal{Y} \to \mathbf{R}$, for every $u \in \mathcal{Y}$

2. We deine the overlap

$$N(r) := \mathrm{esssup}_{x \in X} \sum_{i=1}^{K(r)} \chi_{E_i'}(x).$$

Main statement

Theorem (main compactness result)

Assume **(A), (G)**, *let* $\{u_n\}_{n\in\mathbf{N}} \subseteq \mathcal{Y}$ *bounded in* $D^p(X,\mu)$ *and such that* $\{a_{E_i}(u_n)\}_{n\in\mathbf{N}}$ *is bounded in* $\mathbf{R}$ *for every fived* r *and* $i = 1,\dots,K(r)$.
Moreover, let a) or b) hold where
a) $1 \le p \le q < \infty$ *and*

$$\lim_{r\to 0} C(r)N(r)^{1/p} = 0;$$

b) $1 \le q < p < \infty$ *and*

$$\lim_{r\to 0} C(r)N(r)^{1/p}K(r)^{1/q-1/p} = 0.$$

Then $\{u_n\}$ *has a subsequence converging in* $L^q(E,\nu)$.

Discussion of Assumptions **(A)**

1. Sets E_i and E_i' are *not required* to be contained in balls with certain radii, but they can often be constructed in such a way. The simplest choice of a covering r-family is $E_i = B(x_i, r_i)$, $E_i' = B(x_i, \lambda_i r_i)$ - concentric ballsfor some $\lambda_i \geq 1$. Then the Poincaré type inequality with dilation λ often follows from the usual Poincaré inequality on X.
2. On slit disk (disk with radius removed) one cannot consider the couple $E_i = B(x_i, r_i)$, $E_i' = B(x_i, \lambda_i r_i)$ but the statement also holds.

Proof of main compactness result (situation $p \leq q$)

1. Step 1. *Estimates for a single function u and fixed r.* For every fixed $0 < r := r_m \leq r_0$, consider the covering r-family $\{E_i, E_i'\}_{i=1}^{K(r)}$. Let $u \in \mathcal{Y}$ be arbitrary, with a p-weak upper gradient g. Then by the local Poincaré inequality

$$\sum_{i=1}^{K(r)} \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \leq C(r)^q \sum_{i=1}^{K(r)} \left(\int_{E_i'} g^p \, d\mu \right)^{q/p}$$

$$\overset{q \geq p}{\leq} C(r)^q \left(N(r) \int_X g^p \, d\mu \right)^{q/p}.$$

Last term can be arbitrary small for small r's.

1. Step 2. *Estimates for a Cauchy sequence* $\{u_n\}$. Let:
   - $\{u_n\}_{n=1}^{\infty} \subset \mathcal{Y}$ bounded in $D^p(X,\mu)$ and and that $\{a_{E_i}(u_n)\}_{n=1}^{\infty}$ is bounded for every E_i,
   - g_n be p-weak upper gradients of u_n, $n = 1, 2, \ldots$, such that the sequence $\{g_n\}_{n=1}^{\infty}$ is bounded in $L^p(X,\mu)$.

   We show that $\{u_n\}_{n=1}^{\infty}$ has a Cauchy subsequence in $L^q(E,\nu)$. For all $m, n \geq 1$,

$$\int_E |u_n - u_m|^q \, d\nu \leq 3^{q-1} \sum_{i=1}^{K(r)} \Big(\int_{E_i} |u_n - a_{E_i}(u_n)|^q \, d\nu + \int_{E_i} |u_m - a_{E_i}(u_m)|^q \, d\nu + \int_{E_i} |a_{E_i}(u_n) - a_{E_i}(u_m)|^q \, d\nu \Big).$$

By Step 1 with u replaced by u_n and u_m, respectively, we can for every $\varepsilon > 0$ choose a sufficiently small $r > 0$ in (3) such that for all $m, n \geq 1$,

$$\int_E |u_n - u_m|^q \, d\nu \leq \varepsilon + 3^{q-1} \sum_{i=1}^{K(r)} |a_{E_i}(u_n) - a_{E_i}(u_m)|^q \nu(E_i). \quad (3)$$

1. Step 3. *We finish the proof.* Let $\varepsilon = \frac{1}{2}$ and r- fixed be as in (3). The sequence $\{a_{E_i}(u_n)\}_{n=1}^{\infty}$ is bounded in $\mathbf{R}$ for every $i = 1, \ldots, K(r)$. Hence, applying the Bolzano–Weierstrass theorem, we can for $\varepsilon = \frac{1}{2}$ and a suitable r-family find a subsequence $\{u_n^{(1)}\}_{n=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that the sequence $\{a_{E_i}(u_n^{(1)})\}_{n=1}^{\infty}$ is convergent for every $i = 1, \ldots, K(r)$ and (3) becomes

$$\int_E |u_n^{(1)} - u_m^{(1)}|^q \, d\nu \leq \frac{1}{2} + 3^{q-1} \sum_{i=1}^{K(r)} |a_{E_i}(u_n^{(1)}) - a_{E_i}(u_m^{(1)})|^q \nu(E_i) < 1$$

for all $m, n \geq 1$. For $\varepsilon = \frac{1}{4}$, we find another r-family, and a subsequence $\{u_n^{(2)}\}_{n=1}^{\infty}$ of $\{u_n^{(1)}\}_{n=1}^{\infty}$ such that

$$\int_E |u_n^{(2)} - u_m^{(2)}|^q \, d\nu \leq \frac{1}{2} \quad \text{when } m, n \geq 1.$$

By the diagonal method $\{u_n^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^q(E, \nu)$.□

Doubling measures and Poincaré inequalities

We assume:

1. there exists $r_0 > 0$ such that the p-Poincaré inequality (2) for μ holds for all balls centred in E and with radius at most r_0,
2. μ and ν are (locally) doubling;
3. ν satisfies the (local) dimension condition:

$$\frac{\nu(B(x,r'))}{\nu(B(x,r))} \le C\left(\frac{r'}{r}\right)^{\sigma}, \quad \text{whenever } x \in E \text{ and } 0 < r' \le r \le r_0, \tag{4}$$

   for some $\sigma > 0$, where C is independent of x, r' and r,
4. $u \in D^p_{\mathrm{loc}}(X,\mu)$,
5. $q > p$ and $0 < r \le r_0$, set

$$\Theta(r) := \sup_{0<\rho\le r} \sup_{x\in E} \frac{\rho\nu(B(x,\rho))^{1/q}}{\mu(B(x,\rho))^{1/p}}. \tag{5}$$

Construction of the Poincaré inequalities

Lemma

Under those and general assumptions we have $u \in L^q_{\mathrm{loc}}(E,\nu)$ *and the following two-weighted Poincaré type inequalities*

$$\left(\int_{B\cap E} |u - u_{B,\mu}|^q \, d\nu\right)^{1/q} \le C\Theta(r)\left(\int_{2\lambda B} g_u^p \, d\mu\right)^{1/p}, \quad (6)$$

$$\left(\int_{B\cap E} |u - u_{B\cap E,\nu}|^q \, d\nu\right)^{1/q} \le C\Theta(r)\left(\int_{2\lambda B} g_u^p \, d\mu\right)^{1/p} \quad (7)$$

hold for all balls $B = B(x,r)$ *with* $x \in E$ *and* $10\lambda r \le r_0$*, where* λ *is the dilation constant in* (2)*.*

Compactness result within doubling measures

Theorem

Assume that $\mathcal{Y}$ is a subset of one of the following spaces:

$$D^p(X,\mu) \cap L^1(E,\nu),\ D^p(X,\mu) \cap L^1(X,\mu),\ N^{1,p}(X,\mu).$$

Then the following are true:

1. *If $\Theta(r) \to 0$ as $r \to 0$, then the identity mapping $\mathcal{Y} \hookrightarrow L^q(E,\nu)$ is a compact embedding.*
2. *If $\Theta(r_0) < \infty$ for some $r_0 > 0$, then the identity mapping $\mathcal{Y} \hookrightarrow L^{q'}(E,\nu)$ is a compact embedding for all $q' < q$.*

Optimality of the condition $\Theta(r) \to 0$

This condition appears to be almost optimal by adapted result of Mazya to the metric setting.

Compactness for nondoubling measures

Recall: if $d\mu = w\,dx$ with $w \in B_p(\Omega)$, i.e. $w^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega, dx)$, then $N^{1,p}(\Omega,\mu) = W^{1,p}(\Omega,\mu)$ and $g_u = |\nabla u|$ a.e.

Theorem (embeddings for domain measure in B_θ)

Let $X = \mathbf{R}^n$, $d\mu = w\,dx$, where $w \in B_\theta$ for some $1 < \theta \le p$. Assume that ν satisfies doubling and dimensuion conditions on $E \subset \mathbf{R}^n$ with some $\sigma > \max\{n - p/\theta, 0\}$. Then the identity mappings

$$D^p(\mathbf{R}^n,\mu) \cap L^1(E,\nu) \hookrightarrow L^q(E,\nu), \quad N^{1,p}(\mathbf{R}^n,\mu) \hookrightarrow L^q(E,\nu)$$

are compact if $q(\theta n - p) < \sigma p$.

Particular example within nondoubling measures

Theorem

Assume that $d\mu = w\,dx$ *and* $d\nu = v\,dx$ *in* $\mathbf{R}^n$*, where* $w, v > 0$ *a.e. are two weights such that* $w, w^{-\alpha}, v^{\beta} \in L^1_{\mathrm{loc}}(\mathbf{R}^n, dx)$ *for some fixed* $\alpha > 0$ *and* $\beta > 1$*. Let* $1 + 1/\alpha \le p < n(1 + 1/\alpha)$ *and assume that* $p \ge n(1/\alpha + 1/\beta)$*. Then the identity mappings*

$$D^p(\mathbf{R}^n, \mu) \cap L^1(E, \nu) \hookrightarrow L^q(E, \nu), \quad N^{1,p}(\mathbf{R}^n, \mu) \hookrightarrow L^q(E, \nu)$$

are compact embeddings for every bounded ν*-measurable set* $E \subset \mathbf{R}^n$ *and for all exponents*

$$q \le \frac{n\alpha p\left(1 - \frac{1}{\beta}\right)}{(n-p)\alpha + n}.$$

Trace type compact embedding theorem

Theorem (abbreviated)

Assume **(G)** *and that*

1. μ *supports a p-Poincaré inequality and* $\frac{\mu(B')}{\mu(B)} \geq C\left(\frac{r'}{r}\right)^s$ *on* X *with some* $s > 0$,
2. $F \subset X$ *is such that the d-dimensional Hausdorff measure* Λ_d, *with* $d > \max\{s-p, 0\}$, *satisfies*
$$C_1 r^d \leq \Lambda_d(F \cap B(x,r)) \leq C_2 r^d \tag{8}$$
for all $x \in F$ *and all* $0 < r \leq r_0$,
3. $\mathcal{Y}$ *is one of the spaces:*
$D^p(X,\mu) \cap L^1(E,\nu)$, $D^p(X,\mu) \cap L^1(X,\mu)$, $N^{1,p}(X,\mu)$.

Then the identity mapping $\mathcal{Y} \hookrightarrow L^q(E,\Lambda_d)$ *is compact if* $q(s-p) < dp$.

Remark

The above result applies to many fractal sets in $\mathbf{R}^n$, as well as we recover the well-known compactness of embedding $N^{1,p}(, dx) \hookrightarrow L^q(E, \Lambda_d)$ for all $q < dp/(n-p)$.

Analysis on uniform domains

Definition

An open set $\Omega \subset X$ is a *uniform domain*, if there is a constant $A \geq 1$ such that for every pair $x, y \in \Omega$ there is a curve γ in Ω connecting x and y, so that its length is at most $Ad(x, y)$ and for all $z \in \gamma$,

$$\operatorname{dist}(z, X \setminus \Omega) \geq A^{-1} \min\{l_{xz}, l_{yz}\}, \tag{9}$$

where l_{xz} and l_{yz} are the lengths of the subcurves of γ connecting z to x and y, respectively.

1. Uniform domains were introduced by Martio and Sarvas in 1979 in the context of quasiconformal mappings;
2. Typical examples of uniform domains: convex sets and bounded Lipschitz domains in $\mathbf{R}^n$, many examples of fractal nature, such as the interior of the von Koch snowflake;
3. In the plane, a bounded domain is uniform if and only if its boundary consists only of isolated points and quasicircles, which in turn is equivalent to it being a Sobolev extension domain. In higher dimensions, uniform domains are closely related to extension domains and (ε, δ)-domains of P. Jones.

Compactness result on uniform domains

Theorem (embeddings for uniform domains)

Assume **(G)** *and that*

1. μ *supports a p-Poincaré inequality and* $\frac{\mu(B(x',r'))}{\mu(B(x,r))} \geq C\left(\frac{r'}{r}\right)^s$ *on* X *with some* $s > 0$, *whenever* $x' \in B(x,r)$, $0 < r' \leq r$,
2. $\Omega \subset X$ *is a bounded uniform domain,*
3. *the* ν *satisfies doubling condition and dimension condition* (4) *on* Ω *with some* $\sigma > \max\{s-p, 0\}$.

Then $D^p(\Omega,\mu) = N^{1,p}(\Omega,\mu)$ *as sets and the identity mappings*

$$D^p(\Omega,\mu) \cap L^1(\Omega,\nu) \hookrightarrow L^q(\Omega,\nu), \quad N^{1,p}(\Omega,\mu) \hookrightarrow L^q(\Omega,\nu)$$

are compact if $q(s-p) < \sigma p$.

Further information within uniform domains

1. W focus also on uniform domains, both with power type weights involving distances from the boundary:

$$d\mu_\alpha := \delta(x)^\alpha \, d\mu(x) \quad \text{and} \quad d\nu_\beta := \delta(x)^\beta \, d\nu(x),$$

2. We obtain boundary trace type compact embeddings, recovering the classical ones as well as the ones on fractals, also in weighted setting.

Parallel discussion was devoted to bounded embeddings. In that case in the assumptions there is one covering net (parameter r is not involved).

The result is basaed on joint work with Jana Björn
Poincaré inequalities and compact embeddings into weighted L^q-spaces.

Thank you!

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