Some aspects of fully measurable grand Lebesgue spaces

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- The spaces $L^{p[\cdot],\delta(\cdot)}$ and $L_w^{p[\cdot],\delta(\cdot)}$
- Integral operators
- Duality
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Grand Lebesgue Space $L^{p)}$

The norm

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}}, \qquad 1 < p < \infty$$

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Some Properties

- $L^{p)}$ is a Banach function space
- $L^{p)}$ is not reflexive for any $p > 1$
- $[L^{p)}]^* \neq L^{p')}$
- $L^p \subset L^{p)} \subset L^{p-\varepsilon}, \qquad 0 < \varepsilon \leq p-1$

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A Generalized Grand Lebesgue Space $L^{p),\theta}$

Capone and Fiorenza (2005)

The norm

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^{\theta} \int_0^1 |f|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}}, \qquad 1 < p < \infty$$

For $\theta = 0$, $\quad L^{p),\theta} = L^p$

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The following continuous imbeddings hold:

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A More Generalized Grand Lebesgue Space $L^{p),\delta}$

Capone, Formica and Giova (2013)

$\delta : (0, p-1) \to [0, +\infty)$, a measurable function

$$\rho_{p),\delta}(f) = \operatorname*{ess\,sup}_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_0^1 f^{p-\varepsilon}\, dt \right)^{\frac{1}{p-\varepsilon}}, \qquad 1 < p < \infty$$

Theorem

Let $1 < p < \infty$ and $\delta : (0, p-1) \to [0, +\infty)$ be a measurable function, not identically zero. The mapping $\rho_{p),\delta}$ is equivalent to a Banach function norm if and only if $\delta \in L^\infty(0, p-1)$.

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Definition

Let $1 < p < \infty$. A function δ left continuous on $(0, p-1)$ is said to belong to the class $\mathcal{B}_p$ if

$$\delta(0+) = 0, \quad 0 < \delta \leq 1, \quad \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \uparrow$$

Definition

Let $1 < p < \infty$ and $\delta \in \mathcal{B}_p$. The grand L^p space with respect to δ is defined by

$$L^{p),\delta} = \left\{ f \ : \|f\|_{p),\delta} = \rho_{p),\delta}(|f|) = \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} < \infty \right\}$$

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Fully mesurable grand Lebesgue space $L^{p[\cdot],\delta(\cdot)}$

Let $p(\cdot)$ be a measurable extended real valued function defined on I such that $p(\cdot) \geq 1$ almost everywhere (a.e.), $\delta \in L^\infty$, $\delta > 0$ a.e. and $0 < \|\delta\|_{L^\infty} \leq 1$. The space $L^{p[\cdot],\delta(\cdot)}$ consists of measurable functions f defined on I for which $\|f\|_{L^{p[\cdot],\delta(\cdot)}} := \rho_{p[\cdot],\delta(\cdot)}(|f|) < \infty$, where

$$\rho_{p[\cdot],\delta(\cdot)}(|f|) = \operatorname*{ess\,sup}_{x \in I} \rho_{p(x)}(\delta(x)|f(\cdot)|)$$

and

$$\rho_{p(x)}(\delta(x)|f(\cdot)|) = \begin{cases} \left(\int_I (\delta(x)|f(t)|)^{p(x)} dt \right)^{\frac{1}{p(x)}} & \text{if } 1 \leq p(x) < \infty; \\ \operatorname*{ess\,sup}_{t \in I} (\delta(x)|f(t)|) & \text{if } p(x) = \infty. \end{cases}$$

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- G. Anatriello and A. Fiorenza, *Fully measurable grand Lebesgue spaces*, J. Math. Anal. Appl., 422 (2015), 783-797.

Fully mesurable weighted grand Lebesgue space $L_w^{p[\cdot],\delta(\cdot)}$

Let $w \in L^\infty$ be a weight. The space $L_w^{p[\cdot],\delta(\cdot)}$ consists of measurable functions f defined on I for which $\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} := \rho_{p[\cdot],\delta(\cdot),w}(|f|) < \infty$, where

$$\rho_{p[\cdot],\delta(\cdot),w}(|f|) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x),w(\cdot)}(\delta(x)|f(\cdot)|)$$

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$$\rho_{p(x),w(\cdot)}(\delta(x)|f(\cdot)|) = \begin{cases} \left(\int_I (\delta(x)|f(t)|)^{p(x)} w(t)dt\right)^{\frac{1}{p(x)}}, & 1 \le p(x) < \infty \\ \operatorname*{ess\,sup}_{t\in I}(\delta(x)|f(t)|w(t)), & p(x) = \infty. \end{cases}$$

Integral operators

- Hardy averaging operator $Af(x) = \frac{1}{x}\int_0^x f(y)\,dy$
- Maximal operator $Mf(x) = \sup_{I \ni x} \frac{1}{|I|}\int_I |f|\,dt$
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Notation

- $\mathcal{M}$:= set of extended real valued measurable functions defined on I
- $\mathcal{M}^+$:= subset of $\mathcal{M}$, consisting of non-negative functions
- $\mathcal{M}_0$:= set of finite a.e. measurable functions defined on I
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- $p_+ := \operatorname*{ess\,sup}_{x \in I} p(x)$

Averaging operator : the $L^{p)}$-case

Theorem. (Fiorenza, Gupta and PJ, Studia Math. 2008)

Let $1 < p < \infty$. There exists a constant $c(p) > 1$, independent of f, such that the inequality

$$\left\| \frac{1}{x} \int_0^x f \right\|_{p)} \leq c(p) \, \|f\|_{p)} ,$$

holds for all non-negative measurable functions f in $[0, 1]$ and $c(p)$ is given by

$$c(p) := \inf_{0<\sigma<p-1} (p-1)\sigma^{-\frac{1}{p-\sigma}} \frac{p-\sigma}{p-\sigma-1}$$

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Theorem

Let $p(\cdot) \in \mathcal{M}$, $p(\cdot) > 1$, a.e. $\delta \in L^\infty(I)$, $\delta > 0$ a.e., $0 < \|\delta\| \leq 1$. Then there exisis a constant $c(p(\cdot), \delta) > 1$ such that the inequality

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holds for all $f \in \mathcal{M}_0^+$.

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Theorem (Arino and Muckenhoupt, Trans. AMS, 1990)

Let $1 \leq p < \infty$ and w be a weight function. The inequality

$$\int_0^\infty [Af(x)]^p w(x)dx \leq C \int_0^\infty f^p(x)w(x)dx$$

holds for all non-negative and non-increasing ($\downarrow$) functions f if and only if $w \in B_p$, i.e.,

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Muckenhoupt, 1972

$$1 < p < \infty, \; \|Mf\|_{p,w} \leq C\|f\|_{p,w} \iff w \in A_p$$

i.e.

$$\sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^{p-1} < \infty$$

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$$\sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^{p-1} < \infty$$

Maximal operator

Theorem (A. Fiorenza, B. Gupta, PJ, Studia Math., 2008)

$M : L_w^{p)} \to L_w^{p)},\ p > 1$ *is bounded if and only if* $w \in A_p$.

In other words

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Hilbert transform

$$Hf(x) = p.v. \int_0^1 \frac{f(t)}{x-t} dt$$

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Let $1 < p < \infty$. TFAE

(a) $\|Hf\|_{L^p_w(I)} \le C\|f\|_{L^p_w(I)}, \quad f \in \mathcal{M}^+$

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Duality

- PJ, M. Singh and A.P. Singh, *Duality of fully measurable grand Lebesgue space*, Trans. A. Razmadze Math. Inst., 171 (2017), 32-47.
- G. Anatriello, M.R. Formica and R. Giova, *Fully measurable small Lebesgue spaces*, J. Math. Anal. Appl., 447(2017), 550-563.

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Some preliminaries

A mapping $\rho : \mathcal{M}_0^+ \to [0, \infty]$ is called a **Banach function norm** if for all $f, g, f_n \in \mathcal{M}_0^+$, $n \in \mathbb{N}$, for all constants $\lambda \geq 0$, and for all measurable subsets $E \subset I$, the following properties hold:

- $\rho(f) = 0$ if and only if $f = 0$ a.e. on I
- $\rho(\lambda f) = \lambda \rho(f)$
- $\rho(f + g) \leq \rho(f) + \rho(g)$
- If $0 \leq g \leq f$ a.e. in I, then $\rho(g) \leq \rho(f)$ (lattice property)
- If $0 \leq f_n \uparrow f$ a.e. in I, then $\rho(f_n) \uparrow \rho(f)$ (Fatou property)
- $\rho(\chi_E) < \infty$
- $\int_E f(t)dt \leq C_E\, \rho(f)$, for some constant $C_E < \infty$.

If ρ is a Banach function norm, then the Banach space

$$X = X(\rho) := \{f \in \mathcal{M}_0 : \rho(|f|) < \infty\}$$

is called a **Banach function space (BFS)** with the norm

$$\|f\|_X := \rho(|f|).$$

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If ρ is a Banach function norm, then its **associate norm** ρ' is defined on $\mathcal{M}_0^+$ by

$$\rho'(g) := \sup_{f \in \mathcal{M}^+,\ \rho(f) \leq 1} \int_I f(t)g(t)dt, \quad g \in \mathcal{M}_0^+.$$

The BFS $X' = X'(\rho')$ determined by ρ' is called the **associate space** of X.

- Every BFS X, coincides with its second associate space X''.
- The Banach space dual X^* of a BFS X, is isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm, i.e., $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \to \emptyset$ a.e.
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Duality of $L^{p[\cdot],\delta(\cdot)}$

Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) \geq 1$ a.e., $\delta \in L^\infty$ and $\delta > 0$ a.e. For $g \in \mathcal{M}_0^+$, $E \subseteq I$ and $|E| > 0$, define

$$\rho'_{p[\cdot]',\delta(\cdot),E}(g) := \inf_{\substack{g = \sum g_k \\ g_k \in \mathcal{M}_0}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in E} \rho_{p(x)'} \left(\frac{1}{\delta(x)} |g_k(\cdot)| \right).$$

In particular, when $E = I$, we write $\rho'_{p[\cdot]',\delta(\cdot),E}$ as $\rho'_{p[\cdot]',\delta(\cdot)}$.

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Definition

For $p(\cdot) \in \mathcal{M}$, $p(\cdot) \geq 1$ a.e., $\delta \in L^\infty$ and $\delta > 0$ a.e. on I, we define the **fully measurable small Lebesgue space** by

$$L^{(p[\cdot]',\delta(\cdot)} := \left\{ g \in \mathcal{M}_0 : \|g\|_{L^{(p[\cdot]',\delta(\cdot)}} = \rho'_{p[\cdot]',\delta(\cdot)}(|g|) < \infty \right\}.$$

- $L^{(p[\cdot]',\delta(\cdot)}$ is a Banach space.
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Let $p(\cdot) \in \mathcal{M}$ *be such that* $p(\cdot) > 1$ *a.e. and continuous. Let* $\delta \in L^\infty$, *continuous and* $\delta > 0$ *a.e. with* $\lim_{x \to 0^+} \delta(x) = 0$. *Then*

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- $\left(L^{(p[\cdot]',\delta(\cdot)}\right)' = L^{p[\cdot],\delta(\cdot)}$
- $L^{(p[\cdot]',\delta(\cdot)}$ *has an absolutely norm.*
- $\left(L^{(p[\cdot]',\delta(\cdot)}\right)^* \simeq \left(L^{(p[\cdot]',\delta(\cdot)}\right)' \simeq L^{p[\cdot],\delta(\cdot)}$.
- $L^{p[\cdot],\delta(\cdot)}$ *does not have absolutely continuous norm.*
- $\left(L^{p[\cdot],\delta(\cdot)}\right)' \not\simeq \left(L^{p[\cdot],\delta(\cdot)}\right)^*$.
- *The spaces* $L^{p[\cdot],\delta(\cdot)}$ *and* $L^{(p[\cdot]',\delta(\cdot)}$ *are not reflexive.*

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- $\left(L^{(p[\cdot]',\delta(\cdot)}\right)' = L^{p[\cdot],\delta(\cdot)}$
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Duality of $L^{p[\cdot],\delta(\cdot)}$

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Extrapolation

Rubio de Francia type result

A notation :

$$\|w\|_{B_p} := \inf\left\{C > 0 : \int_0^r w(x)\,dx + \int_r^\infty \left(\frac{r}{x}\right)^p w(x)\,dx \le C\int_0^r w(x),\ r > 0\right\}.$$

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Theorem. (Carro and Lorente, PAMS, 2010)

Let $\eta \geq 0$ *be* $\uparrow$ *on* $(0,\infty)$, $f, g \geq 0$ *be* $\downarrow$ *on* $(0,\infty)$ *and let* $0 < p_0 < \infty$. *Suppose that for every* $w \in B_{p_0}$

$$\int_0^\infty f^{p_0}(x)\, w(x)\, dx \leq \eta(\|w\|_{B_{p_0}}) \int_0^\infty g^{p_0}(x)\, w(x)\, dx.$$

Then, for every $p > 0$ *and* $w \in B_p$

$$\int_0^\infty f^{p}(x)\, w(x)\, dx \leq \tilde{\eta}(\|w\|_{B_{p}}) \int_0^\infty g^{p}(x)\, w(x)\, dx,$$

$$\tilde{\eta}(\|w\|_{B_p}) = \inf_{0<\varepsilon<\frac{p_0}{p\alpha^p\|w\|_{B_p}}} \eta^{p/p_0}\left(\frac{p_0}{\varepsilon}\right)\left(\frac{C\|w\|_{B_p}}{1-\frac{\varepsilon p\alpha^p}{p_0}\|w\|_{B_p}}\right)$$

with $C > 0$ *and* $0 < \alpha < 1$ *being the universal constants.*

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Rubio de Francia type result

Theorem. PJ, M. Singh, A.P. Singh, 2017

Let ψ be a nonnegative nondecreasing function defined on I. Let $1 < s_0 < \infty$. Let (f, g) be a pair of nonnegative, nonincreasing functions on I. Suppose that for every $w \in B_{s_0}$

$$\int_I f^{s_0}(t)w(t)dt \leq \psi(\|w\|_{B_{s_0}}) \int_I g^{s_0}(t)w(t)dt.$$

Then for every $p(\cdot) \in \mathcal{M}$, $p(\cdot) \geq 1$ a.e. and $w \in B_{p_+}$ where $p_+ < \infty$, the inequality

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \leq C\|g\|_{L_w^{p[\cdot],\delta(\cdot)}}$$

holds, where C is a positive constant depending on $p(\cdot)$, s_0, δ and w.

Rubio de Francia type result

The case when $p_+ = \infty$.

Denote

$$B_\infty := \bigcup_{x \in p^{-1}([1,p_+])} B_{p(x)},$$

and

$$\|w\|_{B_\infty} := \inf\left\{\|w\|_{B_{p(x)}} : w \in B_{p(x)}\right\}.$$

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In the framework of Lebesgue spaces, the following result is known:

Theorem. Carro, Lorente, 2009

Let ψ be a nonnegative nondecreasing function defined on I. Let $0 < s_0 < \infty$. Let (f, g) be a pair of nonnegative, nonincreasing functions on I. Suppose for every $w \in B_\infty$

$$\int_I f^{s_0}(t)w(t)dt \leq \psi(\|w\|_{B_\infty}) \int_I g^{s_0}(t)w(t)dt,$$

then for every $s > 0$ a.e. and $w \in B_\infty$, the inequality

$$\left(\int_I f^s(t)w(t)dt\right)^{1/s} \leq (\psi(1))^{1/s_0} \|w\|_{B_\infty}^{1/s} \left(\int_I g^s(t)w(t)dt\right)^{1/s}$$

holds.

Rubio de Francia type result

Theorem. PJ, M. Singh, A.P. Singh, 2017

Let ψ be a nonnegative nondecreasing function defined on I. Let $1 \leq s_0 < \infty$. Let (f, g) be a pair of nonnegative, nonincreasing functions on I. Suppose for every $w \in B_{s_0}$

$$\int_I f^{s_0}(t)w(t)dt \leq \psi(\|w\|_{B_{s_0}}) \int_I g^{s_0}(t)w(t)dt \tag{2}$$

then for every $p(\cdot) \in \mathcal{M}$, $p(\cdot) \geq 1$ a.e. and $w \in B_{p_+}$ where $p_+ = \infty$, the inequality

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \leq C\|g\|_{L_w^{p[\cdot],\delta(\cdot)}} \tag{3}$$

holds, where C is a positive constant depending on $p(\cdot)$, δ, s_0 and w.