Kondratiev spaces from the point of view of Function space theory

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Motivation for yet another scale of Sobolev-type spaces

- Jerison/Kenig ('95): If $D \subset \mathbb{R}^d$ is a Lipschitz domain, then the solution to

$$-\Delta u = f, \qquad u\big|_{\partial D} = 0$$

generally only belongs to $H^{3/2}(D)$, even for $f \in C^\infty(D)$.

- Kondratiev ('67), Grisward ('84): If $D \subset \mathbb{R}^2$ is a polygon, then for $f \in H^s(D)$, $s \geq 0$, the solution to Poisson's equation can be written

$$u = u_{\mathrm{reg}} + u_{\mathrm{sing}},$$

where $u_{\mathrm{reg}} \in H^{s+2}(D)$, and u_{sing} is a finite linear combination of explicitly known singularity functions.

Weighted Sobolev spaces

On domains with polygonal or polyhedral structure the Babuska-Kondratiev spaces $\mathcal{K}^m_{a,p}(D)$ are defined via the norm

$$\|u|\mathcal{K}^m_{a,p}(D)\|^p = \sum_{|\alpha|\leq m} \int_D |\rho(x)^{|\alpha|-a}\partial^\alpha u(x)|^p\, dx\,,$$

where $a \in \mathbb{R}$ is a parameter, and the weight function $\rho : D \longrightarrow [0,1]$ is the smooth distance to the singular set of D. This means ρ is a smooth function, and in the vicinity of the singular set it is equivalent to the distance to that set.

The definition can be extended to domains of the form $D = \mathbb{R}^d \setminus S$, where $S \subset \mathbb{R}^d$ is an arbitrary given closed set. Then again ρ is the smooth distance to S.

Shift theorem for elliptic operators

Proposition

Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, $d = 2, 3$. Consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,, \qquad (1)$$

where $A = (a_{i,j})_{i,j=1}^{d}$ is symmetric and

$$a_{i,j} \in \mathcal{K}^{m}_{0,\infty} = \left\{v : D \longrightarrow \mathbb{C} : \rho^{|\alpha|}\partial^{\alpha} v \in L_{\infty}(D)\,, |\alpha| \le m\right\}.$$

Let the bilinear form

$$B(v, w) = \int_D \sum_{i,j} a_{i,j}(x)\partial_i v(x)\partial_j w(x)dx$$

be bounded and coercive in $H_0^1(D)$.

Shift theorem (continued)

Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$ and any $f \in \mathcal{K}^{m-1}_{a-1,2}(D)$ the problem (1) admits a uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1,2}(D)$, and it holds

$$\|u|\mathcal{K}^{m+1}_{a+1,2}(D)\| \leq C\,\|f|\mathcal{K}^{m-1}_{a-1,2}(D)\|$$

for some constant $C > 0$ independent of f.

A first localization principle

Theorem

Let $D \subset \mathbb{R}^3$ be a bounded polyhedron. Then there exists an open cover

$$D = \bigcup_{j=1}^{J} U_j \cup \bigcup_{k=1}^{K} V_k \cup \bigcup_{l=1}^{L} W_l,$$

such that

- $\rho(x) \geq c$ for all $x \in W_l$, $l = 1, \dots, L$,
- for each j we find exactly one of the vertices $x_r \in \partial D$, such that on U_j we have $\rho(x) \sim |x - x_r|$
- for each k we find exactly one of the edges $\Gamma_s \subset \partial D$, such that on V_k we have $\rho(x) \sim \text{dist}(x, \Gamma_s)$.

A first localization principle II

Let $(\varphi_j)_{j=1}^{J+K+L}$ a smooth partition of unity subordinate to that open cover of D. Then

$$\|u|\mathcal{K}^m_{a,p}(D)\|^p \sim \sum_{j=1}^{J} \|\varphi_j u|\mathcal{K}^m_{a,p}(U_j)\|^p + \sum_{k=1}^{K} \|\varphi_{J+k} u|\mathcal{K}^m_{a,p}(V_k)\|^p + \sum_{l=1}^{L} \|\varphi_{J+K+l} u|\mathcal{K}^m_{a,p}(W_l)\|^p.$$

Main advantage: After translation and rotation, each of the open sets U_j, V_k and W_l is a subdomain of a “standard situation”, i.e. either the unweighted situation, a smooth cone (vertex in $\{0\}$ as the only singular point), or a (smooth or polyhedral) cone with weight given via the distance to the axis.

A first localization principle III

Why is this relevant? One exemplary result:

Theorem

Let $\mathfrak{E}$ be the extension operator as constructed in (Stein 1970). Let $D \subset \mathbb{R}^d$ be a polyhedral Lipschitz domain with singular set S. Then

$$\mathfrak{E} : \mathcal{K}^m_{a,p}(D) \longrightarrow \mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus S).$$

Proof: First show it for one of the standard situation (explicit expression for the weight function is needed), then piece it together for general D.

Second Localization principle for Kondratiev spaces

Theorem

Let $D \subset \mathbb{R}^d$ be a polyhedral domain, and let

$$D_j = \{x \in \overline{D} : 2^{j-1} < |x| < 2^{j+1}\}, \quad j \in \mathbb{Z}\,.$$

Moreover, let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be a resolution of unity w.r.t. $\{D_j\}$, i.e. C^∞-functions with support in D_j and $\sum_j \varphi_j(x) = 1$ for all $x \in \overline{K} \setminus \{0\}$. Additionally assume $\partial^\alpha \varphi_j| \le 2^{j|\alpha|}$ for all $|\alpha| \le m$. Then it holds

$$\|u|\mathcal{K}^m_{a,p}(D)\|^p \sim \sum_{j\in\mathbb{Z}} \|\varphi_j u|\mathcal{K}^m_{a,p}(D)\|^p\,.$$

Main feature: On D_j the distance function ρ is essentially constant.

Refined localization spaces

Definition

Let $\{\varphi_{j,l}\}$ be a resolution of unity of non-negative C^∞-functions w.r.t. a Whitney decomposition $\{Q_{j,k_l}\}$, i.e.

$$\sum_{j,l} \varphi_{j,l}(x) = 1 \text{ for all } x \in D\,, \qquad |\partial^\alpha \varphi_{j,l}(x)| \le c_\alpha 2^{j|\alpha|}\,, \quad \alpha \in \mathbb{N}_0^d\,.$$

Moreover, we require $\operatorname{supp} \varphi_{j,l} \subset 2Q_{j,k_l}$. Then we define the refined localization spaces $F^{s,\mathrm{rloc}}_{p,q}(D)$ to be the collection of all locally integrable functions f such that

$$\|f|F^{s,\mathrm{rloc}}_{p,q}(D)\| = \left(\sum_{j=0}^{\infty} \sum_{l=1}^{K_j} \|\varphi_{j,l} f | F^s_{p,q}(\mathbb{R}^d)\|^p \right)^{1/p} < \infty\,.$$

Equivalent descriptions:

(i) If D is a Lipschitz domain and $s > \min(0, d(\frac{1}{\min(p,q)} - 1))$, then

$$F^{s,\mathrm{rloc}}_{p,q}(D) = \{f \in F^s_{p,q}(\mathbb{R}^d) : \operatorname{supp} f \subset \overline{D}\}\,.$$

(ii) For arbitrary domains holds

$$\|u|F^{s,\mathrm{rloc}}_{p,q}(D)\| \sim \|u|F^s_{p,q}(D)\| + \|\delta(\cdot)^{-s}u|L_p(D)\|\,,$$

where $\delta(x) = \mathrm{dist}(x, \partial D)$ is the distance to the boundary.

Prototypical domain when investigating function spaces on polygons or polyhedra:

$$D_\ell = \mathbb{R}^d \setminus \mathbb{R}^\ell\,, \quad 0 \le \ell \le d-1\,.$$

Theorem

Let $m \in \mathbb{N}$, $1 < p < \infty$ and $0 \le \ell < d$. Then it holds

$$\mathcal{K}^m_{m,p}(D_\ell) = F^{m,\mathrm{rloc}}_{p,2}(D_\ell)\,.$$

Corollary

For all $a \ge 0$ we have

$$\|u|\mathcal{K}^m_{a,p}(D_\ell)\| \sim \sum_{|\alpha|=m} \|\partial^\alpha(\rho^{m-a}u)|L_p(\mathbb{R}^d)\| + \|\rho^{-a}u|L_p(\mathbb{R}^d)\|\,.$$

One application of refined localization spaces

Proposition

Let $D \subset \mathbb{R}^3$ be a polyhedral domain, and let $m \in \mathbb{N}$, $a \in \mathbb{R}$. Then the embedding

$$\mathcal{K}^m_{a,p}(D) \hookrightarrow \mathcal{K}^{m-1}_{a-1,p}(D)$$

is compact.

Idea of the proof: The embedding $F^{m,\mathrm{rloc}}_{p,2}(D) \hookrightarrow F^{m-1,\mathrm{rloc}}_{p,2}(D)$ is compact for arbitrary domains (in turn, this follows, e.g., from their wavelet characterization). Combine this with a shifting operator for the weight function.

Kondratiev spaces of fractional smoothness

Standard approach: Introduce fractional orders of smoothness via complex interpolation

$$\mathcal{K}^s_{a,p}(D) = \left[\mathcal{K}^m_{a,p}(D), \mathcal{K}^{m+1}_{a,p}(D)\right]_\Theta$$

for $s = m + \Theta$.
Problem: No unified results for all parameters $s > 0$, no easy argument for

$$\mathcal{K}^m_{a,p}(D) = \left[\mathcal{K}^{m-1}_{a,p}(D), \mathcal{K}^{m+1}_{a,p}(D)\right]_{1/2}.$$

New definition

New approach: exploit connection to refined localization spaces (which can be easily seen to be a scale of spaces closed under complex interpolation).

Definition

Let $s \in \mathbb{R}$, $1 < p < \infty$ and $a \in \mathbb{R}$. Then we define

$$\mathcal{K}^s_{s,p}(\mathbb{R}^d \setminus S) = F^{s,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S)$$

as well as

$$\mathcal{K}^s_{a,p}(\mathbb{R}^d \setminus S) = T_{a-s} F^{s,\mathrm{rloc}}_{p,2}(\mathbb{R}^d \setminus S).$$

Here $T_b : u \mapsto \rho^b u$ is the canonical isomorphism

$$T_b : \mathcal{K}^m_{a,p}(D) \longrightarrow \mathcal{K}^m_{a+b,p}(D).$$

Lemma

Let $s \in \mathbb{R}_+$, $a \in \mathbb{R}$, *and let* $m_1, m_2 \geq 0$ *be integers such that* $s = (1-\Theta)m_1 + \Theta m_2$. *Then we have*

$$\mathcal{K}^s_{a,p}(\mathbb{R}^d \setminus S) = \left[\mathcal{K}^{m_1}_{m_1-s+a,p}(\mathbb{R}^d \setminus S), \mathcal{K}^{m_2}_{m_2-s+a,p}(\mathbb{R}^d \setminus S)\right]_\Theta$$

The boundedness of Stein's extension operator further ensures that everywhere we may replace $R^d \setminus S$ by a polyhedral Lipschitz domain D.

The interpolation result in full generality

Theorem

Let $s, s_0, s_1 \in \mathbb{R}_+$, $1 < p, p_0, p_1 < \infty$ and $a, a_0, a_1 \in \mathbb{R}$, such that for some $\Theta \in (0,1)$

$$s = (1-\Theta)s_0 + \Theta s_1, \quad a = (1-\Theta)a_0 + \Theta a_1, \quad \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

Then it holds

$$\left[\mathcal{K}^{s_0}_{a_0,p_0}(D), \mathcal{K}^{s_1}_{a_1,p_1}(D)\right]_\Theta = \mathcal{K}^{s}_{a,p}(D).$$

An application: Sobolev-type embedding

Theorem

Let D be a polyhedral Lipschitz domain, and let

$$s_0 - \frac{d}{p_0} \geq s_1 - \frac{d}{p_1}, \qquad a_0 - \frac{d}{p_0} \geq a_1 - \frac{d}{p_1}.$$

Then we have a continuous embedding

$$\mathcal{K}^{s_0}_{a_0,p_0}(D) \hookrightarrow \mathcal{K}^{s_1}_{a_1,p_1}(D).$$

Moreover, both conditions are also necessary.

Thank you for your attention