Compactness in spaces of functions of bounded variation

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Functions of bounded Jordan variation

Definition (Jordan, 1881)

Let $x : [0,1] \to \mathbb{R}$. The number

$$\bigvee_0^1(x) = \sup \sum_{i=1}^{N} |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_N = 1$ of the interval $[0,1]$ is said to be the variation (in the sense of Jordan) of the function x.

Remark

We will further write $|x(J)| = |x(b) - x(a)|$ *for an interval* $J = [a, b]$.

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6. etc...

Different generalizations

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- Λ-variation....

Λ-variation

Definition

Let us consider a nondecreasing sequence of positive real numbers $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$. We call such sequence a *Waterman sequence* if $\sum_{n=1}^{\infty} 1/\lambda_n = +\infty$. Waterman sequence is called *proper* if $\lambda_n \to +\infty$.

Definition (Waterman, 1972)

Let Λ be a Waterman sequence and let $x\colon I \to \mathbb{R}$. We say that x is of bounded Λ-variation if there exists a positive constant M such that for any finite sequence of nonoverlapping subintervals $\{I_1, \dots, I_n\}$ of I, the following inequality holds

$$\sum_{k=1}^{n} \frac{|x(I_k)|}{\lambda_k} \leq M.$$

The supremum of the above sums taken over the family of all the finite collections of nonoverlapping subintervals of I is called the Λ-variation of x and it is denoted by $\mathrm{var}_\Lambda(x)$. The set of all functions satisfying $\mathrm{var}_\Lambda(x) < +\infty$ will be denoted as ΛBV.

Why Λ-variation?

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- Origins: Waterman's result of Fourier series;

Theorem (Waterman, 1972)

If $x\colon I \to \mathbb{R}$ is of bounded Λ-variation for $\Lambda = (1/n)_{n\in\mathbb{N}}$ (so called harmonic bounded variation), then the Fourier series of x converge everywhere and converge uniformly on closed intervals of continuity of x. No larger class of ΛBV functions has this property, i.e. in a larger ΛBV space there exists such function that its Fourier series diverges at a point.

- Perlman's result on regulated functions.

Theorem (Perlman, 1980)

For proper Waterman sequences Λ, Λ_n:

a) $\bigcap_{\Lambda} \Lambda BV = BV$;

b) $\bigcup_{\Lambda} \Lambda BV = \mathcal{R}$, *where $\mathcal{R}$ is the set of all bounded regulated functions;*

c) *for any sequence of proper Waterman sequences Λ_n we have* $\bigcap_{n\in\mathbb{N}} \Lambda_n BV \neq BV$;

d) *for any sequence of proper Waterman sequences Λ_n we have* $\bigcup_{n\in\mathbb{N}} \Lambda_n BV \neq \mathcal{R}$;

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- not reflexive (Prus-Wiśniowski, 2012);
- Helly's selection theorem holds: if $\|x\|_\Lambda \leq M$, for $(x_n) \subset \Lambda BV$ and $M \geq 0$, then there exists such pointwise convergent subsequence x_{n_k}, than $x_{n_k} \to x_0$ and $\|x_0\|_\Lambda \leq M$;

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- norm stronger then supremum norm, i.e. there exists constant $c_\Lambda > 0$ such that $\|x\|_\infty \leq c_\Lambda \|x\|_\Lambda$.

Hierarchy of ΛBV spaces

Theorem (Perlman, Waterman, 1979)

Let Λ and Γ be Waterman's sequences. Then:

(i)

$$\Lambda BV \subseteq \Gamma BV \Leftrightarrow \sum_{k=1}^{n} 1/\gamma_k = O(\sum_{k=1}^{n} 1/\lambda_k);$$

(ii)

$$\Lambda BV = \Gamma BV \Leftrightarrow \exists_{c,c' \in (0,+\infty)} c \leq \frac{\sum_{k=1}^{n} 1/\gamma_k}{\sum_{k=1}^{n} 1/\lambda_k} \leq c'.$$

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Remark

If $1/\gamma_n = O(1/\lambda_n)$ then, $\sum_{k=1}^{n} 1/\gamma_k = O(\sum_{k=1}^{n} 1/\lambda_k)$. This condition may also be interpreted as $\lambda_n = O(\gamma_n)$.

Compact embeddings

Theorem (D. Bugajewski, J.G., P. Kasprzak, 2016)

Let Λ and Γ be Waterman sequences such that $1/\gamma_n = o(1/\lambda_n)$ (i.e. $\lambda_n = o(\gamma_n)$). Then if $A \subset \Lambda BV$ is bounded and compact in $\|\cdot\|_\infty$ norm, then A is compact subset of ΓBV.

Compact embeddings – proof

Proof (an idea).

- each sequence $(x_n) \subset A$ contains subsequence (x_{n_k}) uniformly convergent to a function x_0; by Helly's selection theorem we may also assume that $x_0 \in \Lambda BV$;
- we know that for $m \geq j$, for j large enough, we have $1/\gamma_m \leq \varepsilon \cdot 1/\lambda_m$;
- now let us take any set of nonoverlapping intervals $\{I_m : m \in \mathbb{N}\}$; then for $k \geq k_0$ for $k_0 \in \mathbb{N}$ large enough we may assume that $|(x_{n_k} - x_0)(I_m)| \leq \varepsilon$, for any interval I_m;
- Then

$$\sum \frac{|(x_{n_k} - x_0)(I_m)|}{\gamma_m} = \sum_{m<j} \frac{|(x_{n_k} - x_0)(I_m)|}{\gamma_m} + \sum_{m \geq j} \frac{|(x_{n_k} - x_0)(I_m)|}{\gamma_m} \leq$$

$$\varepsilon \sum_{m<j} \frac{1}{\gamma_m} + \varepsilon \sum_{m \geq j} \frac{|(x_{n_k} - x_0)(I_m)|}{\lambda_m} \leq \varepsilon \sum_{m<j} \frac{1}{\gamma_m} + \varepsilon \left(\mathrm{var}_\Lambda(x_0) + \mathrm{var}_\Lambda(x_{n_k})\right).$$

□

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It is an easy observation that if $x \in \Lambda BV \subset \Gamma BV$ where $\Lambda = o(\Gamma)$, then

$$\lim_{m \to +\infty} \mathrm{var}_{\Gamma_{(m)}}(x) = 0,$$

where $\Gamma_{(m)} = (\gamma_n)_{n \geq m}$. Such functions are known as *continuous in Γ-variation* (Waterman, 1977).

The space of all functions continuous in Λ-variation will be denoted as ΛBV_c.

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Theorem (F. Prus-Wiśniowski, 2008)

The following conditions are equivalent:

- *the space* $C\Lambda BV = C[0,1] \cap \Lambda BV$ *is separable;*
- $C\Lambda BV_c = C\Lambda BV$;
- $\Lambda BV_c = \Lambda BV$;
- $S_\Lambda < 2$ *where*

$$S_\Lambda = \limsup_{n \to +\infty} \frac{\sum_{i=1}^{2n} 1/\lambda_i}{\sum_{i=1}^{n} 1/\lambda_i},$$

is called the Shao-Sablin index of Λ.

Shao-Sablin index

Remark

If $\lim_{n\to+\infty} \frac{\lambda_{2n}}{\lambda_n} = \alpha$, *then* $S_\Lambda = \frac{2}{\alpha}$.

Example

If $\lambda_n = n$, then $S_\Lambda = 1$. If $\lambda_n = \ln n$, then $S_\Lambda = 2$.

Necessary condition for compactness when $S_\Gamma < 2$

Theorem (D. Bugajewski, K. Czudek, J. G., J. Sadowski, 2017)

If $A \subset \Gamma BV_c$ is compact in ΓBV, then there exists such $\Lambda = o(\Gamma)$ that $A \subset \Lambda BV$ and A is bounded in ΛBV.

Appendix: relation to ΦBV spaces

Assume $\phi\colon [0,+\infty) \to [0,+\infty)$ is convex, increasing, $\phi(0) = 0$, $\lim_{s\to 0+} \phi(s)/s = 0$ and $\lim_{s\to +\infty} \phi(s)/s = +\infty$. Let $\psi\colon [0,+\infty) \to [0,+\infty)$ be the convex conjugate of ϕ, i.e. $\psi(x) = \sup_{y>0}\{xy - \phi(y)\}$.

Theorem (Y. Ge, H. Wang, 2015)

1. *The inclusion $\Phi BV \subset \Lambda BV$ holds iff there exists a constant $c > 0$ such that*
$$\sum_{n=1}^{+\infty} \psi\Big(\frac{1}{c\lambda_n}\Big) < +\infty.$$
2. *The inclusion $\Lambda BV \subset \Phi BV$ holds iff there exists a constant $c > 0$ such that*
$$\sup_{1\le k<+\infty} k\phi\Big(c\Big(\sum_{j=1}^{k} 1/\lambda_j\Big)^{-1}\Big) < +\infty.$$