The embeddings of Jawerth and Franke for Besov and Triebel-Lizorkin spaces with variable exponents

Helena Gonçalves[1]

Chemnitz University of Technology, Germany

New perspectives in the theory of function spaces and their applications

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[1] joint work with H. Kempka and J. Vybíral

Theorem - Sobolev embeddings

Let $-\infty < s_1 < s_0 < \infty$ and $0 < p_0 < p_1 \leq \infty$ with $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.

(i) If $0 < q_0 \leq q_1 \leq \infty$, then

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^n).$$

(ii) If $0 < q_0, q_1 \leq \infty$ and $p_1 < \infty$, then

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^n).$$

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$\Rightarrow$ These results are not optimal!

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- (Vy08) J. Vybíral, A new proof of the Jawerth-Franke embedding. Rev. Mat. Complut., **21**, 75–82 (2008).

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  2. use of duality and non-increasing rearrangements theory

- (GKV) H. F. Gonçalves, H. Kempka, J. Vybíral, Franke-Jawerth embeddings for Besov and Triebel-Lizorkin spaces with variable exponents. Ann. Acad. Sci. Fenn. Math., **43**, 1–23 (2018).

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Class of variable exponents

- $\mathcal{P}(\mathbb{R}^n) = \{p : \mathbb{R}^n \to (0,\infty]$ mesurable, bounded away from $0\}$
- $\mathbb{R}^n_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\}$
- $p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p^+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)$

Definition

For $p \in \mathcal{P}(\mathbb{R}^n)$, we define the **modular**

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n\setminus\mathbb{R}^n_\infty} |f(x)|^{p(x)} \mathrm{d}x + \operatorname*{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|$$

and the **variable exponent Lebesgue space** $L_{p(\cdot)}(\mathbb{R}^n)$ by

$$L_{p(\cdot)}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \,|\, \varrho_{p(\cdot)}(f/\lambda) < \infty, \text{ for some } \lambda > 0\}.$$

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For $p \in \mathcal{P}(\mathbb{R}^n)$, the space $L_{p(\cdot)}(\mathbb{R}^n)$ is a quasi-Banach space with the Luxembourg norm

$$\|f \mid L_{p(\cdot)}(\mathbb{R}^n)\| := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(f/\lambda\right) \leq 1\right\}.$$

(KR91) O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*. Czechoslovak Math. J., **40**(116), no. 4, 592–618 (1991).

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Regularity conditions

Definition

Let $g \in C(\mathbb{R}^n)$.

(i) g is *locally* $\log$-*Hölder continuous*, $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$\exists c_{\log} > 0 : \quad |g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x-y|)}, \quad \forall x, y \in \mathbb{R}^n.$$

(ii) g is *globally* $\log$-*Hölder continuous*, $g \in C^{\log}(\mathbb{R}^n)$ if $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and

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Smooth dyadic resolution of unity

- Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$:
  (i) $\varphi_0(x) = 1$ if $|x| \leq 1$
  (ii) $\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$
- We define $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) := \varphi(2^{-j}x), \forall j \in \mathbb{N}$. Then,

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

The sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ forms a *smooth dyadic resolution of unity*.

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$$\|(f_j)_{j\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\| = \left\| \left(\sum_{j=0}^{\infty} |f_j(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \mid L_{p(\cdot)}(\mathbb{R}^n) \right\|$$

B-spaces

(AH10) A. Almeida, P. Hästö, Besov spaces with variable smoothness and integrability. J. Funct. Anal., **258**, no. 5, 1628–1655 (2010).

Modular: $$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}((f_j)_{j\in\mathbb{N}_0}) = \sum_{j=0}^{\infty} \inf \left\{ \lambda_j > 0 : \varrho_{p(\cdot)}\left(\frac{f_j}{\lambda_j^{1/q(\cdot)}} \right) \le 1 \right\}$$

Norm: $$\|(f_j)_{j\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| = \inf \left\{ \mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_j}{\mu} \right) \le 1 \right\}$$

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$$\|f \mid B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_\varphi := \left\| \left(2^{js(\cdot)} \, (\varphi_j \widehat{f})^\vee\right)_{j\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\right\| < \infty.$$

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Let $s \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$. Then $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_\varphi := \left\| \left(2^{js(\cdot)} \, (\varphi_j \widehat{f})^\vee\right)_{j\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\right\| < \infty.$$

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- Independence of the chosen resolution of unity $(\varphi_j)_{j\in\mathbb{N}_0}$
- Characterizations by smooth and non-smooth atoms, local means, molecules, ...
- Sobolev embedding results
- Pointwise multipliers assertion
- Trace results

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Decomposition of function spaces

Idea

Represent functions as linear combination of basic functions

$$f \in A^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

where...

$\lambda_{\nu m}$ $\rightsquigarrow$ elements of the sequence space $a^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

$a_{\nu m}$ $\rightsquigarrow$ building blocks (we chose atoms).

Moreover,

$$\|f \mid A^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid a^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|.$$

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Sobolev embedding

Theorem [AH10]

Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $s_0, s_1 \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$. Let $s_0(x) \geq s_1(x)$ and $p_0(x) \leq p_1(x)$ for all $x \in \mathbb{R}^n$ with

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Then we have

$$b^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow b^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Jawerth embedding - sequence spaces

Theorem

Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_0^+ < \infty$ and $s_0, s_1 \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$. Let

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) > 0 \quad \text{with} \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Then

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The proof I

Conditions

$$\inf_{x\in\mathbb{R}^n}(s_0(x)-s_1(x))>0 \qquad \text{with} \qquad s_0(x)-\frac{n}{p_0(x)}=s_1(x)-\frac{n}{p_1(x)}, \quad x\in\mathbb{R}^n$$

1. Let

$$\varepsilon' := \inf_{x\in\mathbb{R}^n}(s_0(x)-s_1(x)) = \inf_{x\in\mathbb{R}^n} n\left(\frac{1}{p_0(x)}-\frac{1}{p_1(x)}\right) > 0.$$

Then,

$$\frac{p_1(x)}{p_0(x)}-1 = p_1(x)\left(\frac{1}{p_0(x)}-\frac{1}{p_1(x)}\right) \geq p_1^-\frac{\varepsilon'}{n}.$$

For $\varepsilon = p_1^-\frac{\varepsilon'}{2n} > 0$ we get

$$p_0(x) < (1+\varepsilon)p_0(x) < p_1(x), \quad \text{for all } x\in\mathbb{R}^n.$$

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$$f^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow b^{s_1(\cdot)}_{p_1(\cdot),p_0(\cdot)}(\mathbb{R}^n)$$

2. By a Sobolev embedding, we have

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$$f^{s_0(\cdot)}_{p_0(\cdot),\infty}(\mathbb{R}^n) \hookrightarrow b^{\tilde{s}(\cdot)}_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}(\mathbb{R}^n)$$

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where $s_0(x) - \tilde{s}(x) = -\dfrac{n}{p_0(x)} + \dfrac{n}{(1+\varepsilon)p_0(x)} = \dfrac{n}{p_0(x)}\dfrac{\varepsilon}{1+\varepsilon}, x \in \mathbb{R}^n$

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5. Equivalent to prove

$$\|\lambda \mid b^0_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}(\mathbb{R}^n)\| \leq C\,\|\lambda \mid f^{\frac{n}{p_0(\cdot)}\cdot\frac{\varepsilon}{1+\varepsilon}}_{p_0(\cdot),\infty}(\mathbb{R}^n)\|,$$

   for some constant $C > 0$ and $\lambda = (\lambda_{j,m})_{j,m}$, for $j \in \mathbb{N}_0, m \in \mathbb{Z}^n$.

6. After calculations, we arrive to the constant exponent case, which was already proved.

Remark

For $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, it holds $\quad 2^{j\frac{1}{p(x)}} \sim 2^{j\frac{1}{p(y)}}, \quad$ for $|x-y| \leq c\,2^{-j}$.

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Jawerth embedding - function spaces

Corollary

Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_0^+, q^+ < \infty$ and $s_0, s_1 \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. Let

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) > 0 \quad \text{with} \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

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Franke embedding

Theorem

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Thank you for your attention!