Modular and norm inequalities for operators on the cone of decreasing functions in Orlicz space

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Abstract. We show that the modular and norm inequalities on the cone of nonnegative decreasing functions in Orlicz space are equivalent to modified modular and norm inequalities on the cone of all nonnegative functions in Orlicz space.

PLAN OF TALK

1. ORLICZ SPACES. PRELIMINARIES.

Introduce class Θ_0 :

$$\Theta_0 := \left\{ \begin{array}{l} \Phi : [0, \infty) \to [0, \infty]; \quad \Phi \uparrow; \quad \Phi(t) = 0 \Leftrightarrow t = 0; \\ \exists\, t_\infty = t_\infty(\Phi) \in (0, \infty]: \\ \Phi \in C\left[0, t_\infty\right), \; \Phi\left(t_\infty - 0\right) = \infty \end{array} \right\}. \qquad (1.1)$$

$$\Phi^{-1}(\tau) = \inf\{t \in [0, \infty): \quad \Phi(t) \geq \tau\}, \qquad \tau \in [0, \infty). \qquad (1.2)$$

$M = M(R_+)$ - set of Lebesgue- measurable functions,

Always we assume that

$$\Phi \in \Theta_0; \qquad v \in M, \; 0 < v < \infty \text{ on } R_+ . \qquad (1.3)$$

For $\lambda > 0,\ f \in M$ we denote

$$J_{\lambda}(f) := \int_0^{\infty} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, dx, \tag{1.4}$$

$$\| f \|_{\Phi, v} = \inf \left\{ \lambda > 0 :\ J_{\lambda}(f) \le 1 \right\}. \tag{1.5}$$

Definition 2.1. *Orlicz space* $L_{\Phi, v}$ *is defined as the set of functions* $f \in M : \| f \|_{\Phi, v} < \infty$.

Let $p \in (0,1]$, Φ be p - convex on $[0, t_{\infty})$, that is for

$\alpha, \beta \in (0,1],\ \alpha^{p} + \beta^{p} = 1,$

$$\Phi(\alpha t + \beta \tau) \le \alpha^{p}\Phi(t) + \beta^{p}\Phi(\tau),\quad t, \tau \in [0, t_{\infty}). \tag{1.6}$$

The following result is essentially known (see, for example [1, 2,]).

Theorem 1.1. *Let* $\Phi \in \Theta_0$; $\quad v \in M,\ 0 < v < \infty$ *, and the condition of* p *- convexity (1.6) be fulfilled. Then,*

1) The triangle inequality takes place in $L_{\Phi,v}$ *:*

if $f, g \in L_{\Phi,v}$*, then* $f + g \in L_{\Phi,v}$ *, and*

$$\| f + g \|_{\Phi,v} \le \left(\| f \|_{\Phi,v}^{p} + \| g \|_{\Phi,v}^{p} \right)^{1/p}.$$

2) $\| f \|_{\Phi,v}$ *is monotone quasi-norm (norm if* $p=1$*) :*

$$f \in M,\ |f| \le g \in L_{\Phi,v} \Rightarrow f \in L_{\Phi,v},\ \| f \|_{\Phi,v} \le \| g \|_{\Phi,v},$$

that has Fatou property:

$$f_n \in M,\ 0 \le f_n \uparrow f \ \Rightarrow \ \| f \|_{\Phi,v} = \lim_{n \to \infty} \| f_n \|_{\Phi,v}. \qquad (1.7)$$

Conclusion. *Under conditions of Theorem 1.1* $L_{\Phi,v}$ *forms an ideal space which is quasi – Banach space (Banach space if* $p=1$*) and has Fatou property.*

Example 1.2. Let $p \in R_{+}$, $\Phi(t) = p^{-1} t^{p}$.

Then, Φ is p_1 - convex with $p_1 = \min\{p,\ 1\}$. We have: $L_{\Phi, v} = L_p(v)$ is Lebesgue space.

Example 1.3. Let $\Phi : [0, \infty) \to [0, \infty)$ be *Young function:*

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau; \quad \varphi(0) = 0,\ \varphi \uparrow\ ;\ 0 < \varphi(t) < \infty,\ t \in R_{+};$$

φ is left-continuous. Then Φ is convex on $[0,\ \infty)$ (see (1.6) with p=1).

For example

$$\begin{cases} \varphi(0)=0,\ \varphi(t)=1,\ t\in(0,1];\ \varphi(t)=e^{t-1},\ t\in(1,\infty) \Rightarrow \\ \qquad \Phi(t)=t,\ t\in[0,1];\ \Phi(t)=e^{t-1},\ t\in(1,\infty) \end{cases}$$

Φ *is called* N*- function* if $\varphi(+0) = 0$, $\varphi(+\infty) = \infty$.

2. RELATIONS BETWEEN MODULAR AND NORM INEQUALITIES

In this Section we show that modular inequality on the cone in Orlicz space is equivalent to one- parametrical family of norm inequalities on this cone.

For function $\Phi \in \Theta_0$ introduce,

$$\sigma_{\Phi}(\varepsilon) = 1/\Phi\left(\varepsilon^{-1}\right), \quad \varepsilon^{-1} \in \left(0,\ t_{\infty}(\Phi)\right). \tag{2.1}$$

Let $K \subset M_{+} \bigcap L_{\Phi,u}$ be some cone of functions and

$T: K \rightarrow M_{+}$ be a positively homogenous operator, that is

$$f \in K,\ \alpha \geq 0 \quad \Rightarrow \quad \alpha f \in K;\ T(\alpha f) = \alpha T(f).$$

Theorem 2.1. *Let* $u, v, w \in M,\ 0 < u,\ v, w < \infty$;

$$\Phi_1, \Phi_2 \in \Theta_0, \quad t_\infty(\Phi_1) = t_\infty(\Phi_2) \equiv t_\infty \ .$$

Then, the modular inequality

$$\Phi_2^{-1}\left\{\int_0^\infty \Phi_2(wTf)v\,dt\right\} \le \Phi_1^{-1}\left\{\int_0^\infty \Phi_1(Cf)u\,dt\right\} \qquad (2.2)$$

holds for all $f \in K$ *with a constant* $C \in R_+$ *not depending of* f *if and only if the following norm inequalities hold for all* $f \in K$ *and* $\varepsilon \in \left(t_\infty^{-1}, \infty\right)$

$$\|wTf\|_{\Phi_2, \sigma_{\Phi_2}(\varepsilon)v} \le C\,\|f\|_{\Phi_1, \sigma_{\Phi_1}(\varepsilon)u} \ . \qquad (2.3)$$

Corollary 2.2. *In the conditions of Theorem 2.1 let* $\Phi_1 = \Phi_2 \equiv \Phi$ *. Then the integral inequality holds*

$$\int_0^\infty \Phi\left(wTf\right) v\,dt \le \int_0^\infty \Phi\left(Cf\right) u\,dt \tag{2.4}$$

for all $f \in K$ *with constant* $C \in R_+$ *not depending of* f *if and only if the norm inequalities hold*

$$\| wTf \|_{\Phi,\delta v} \le C \| f \|_{\Phi,\delta u} \tag{2.5}$$

for all $f \in K$ *and* $\delta \in R_+$.

3. REDUCTION THEOREM FOR NORMS OF OPERATORS ON THE CONE OF MONOTONE FUNCTIONS.

$\Phi : [0, \infty) \to [0, \infty)$ be Young function, see Example 1.3:

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau; \quad \varphi(0) = 0, \ \varphi \uparrow \ ; \ 0 < \varphi(t) < \infty, \ t \in R_+.$$

Ψ be complementary Young function for Φ, that is

$$\Psi(t) = \int_0^t \psi(\tau) d\tau, \quad t \in [0, \infty); \quad \psi(\tau) = \inf \{\sigma: \quad \varphi(\sigma) \geq \tau\}$$

(the complementary properties are mutual).

Example 3.1.

$p > 1, \ 1/p + 1/p' = 1; \quad \varphi(t) = t^{p-1}, \ \Phi(t) = p^{-1} t^p,$

$\psi(\tau) = t^{p'-1}, \ \Psi(t) = p'^{-1} t^{p'}.$

We consider the cone of nonnegative monotone functions

$$\Omega = \left\{ f \in L_{\Phi, v} : \quad 0 \leq f \downarrow \right\}. \tag{3.1}$$

For $g \in M_+$ the *associated norm on the cone* Ω

$$\| g \|'_{\Omega} := \sup \left\{ \int_0^\infty f\, g\, dt : \quad f \in \Omega ;\ \| f \|_{\Phi, v} \le 1 \right\}. \qquad (3.2)$$

Theorem 3.2. *Let* Φ, Ψ *be the complementary Young functions,*

$v \in M,\ 0 < v < \infty,\quad V(t) = \int_0^t v\, d\tau < \infty,\ t \in R_+,\quad V(\infty) = \infty,$

$a \in (0,1)$ *be fixed. The following two-sided estimate holds for the associate norm (3.2) with constants depending on* $a \in (0,1)$

$$\| g \|'_{\Omega} \cong \left\| \Re_a (g; t) \right\|_{\Psi, v} . \qquad (3.3)$$

Here,

$$\Re_a (g; t) := V(t)^{-1} \int_{\delta_a(t)}^t g(\tau)\, d\tau , \quad . \qquad (3.4)$$

$$\delta_a(t) := V^{-1}(a V(t)),\ t \in R_+ . \qquad (3.5)$$

Remark 3.3. *Let us assume additionally that function* Φ *in Theorem 3.2 satisfies* Δ_2*-condition, that is*

$$\exists C \in (1, \infty): \quad \Phi(2t) \le C\,\Phi(t), \ \forall t \in R_+ \qquad (3.6)$$

Then we can set $a=0$ *in (3.3), and (3.4) so that*

$$\|g\|'_{\Omega} \cong \left\| V(t)^{-1} \int_0^t g \, d\tau \right\|_{\Psi, v} \qquad (3.7)$$

with constants depending only on the constant C *in* (3.6).

We will use the formula for the conjugate operator to the operator (3.4):

$$\Re_a^*(f;\tau) = \int_\tau^{\delta_a^{-1}(\tau)} \frac{f(t)}{V(t)}\, dt, \ \tau \in R_+.$$

Theorem 3.4. *Let* T, T^* *be positively homogeneous conjugate operators:*

$$\int\limits_{R_+} g\,T f\, d\tau = \int\limits_{R_+} f\, T^* g\, d\tau\ , \quad f, g \in M_+ .$$

Let Φ_1, Φ_2 *be Young functions, and* Ψ_1, Ψ_2 *be their complementary functions. Let* $u, v, w \in M$, $0 < u,\ v, w < \infty$; *and the conditions of Theorem 3.2 on weight* v *be fulfilled. Then the following three inequalities are equivalent:*

$$\left\| wT f \right\|_{\Phi_2, u} \le c_1 \left\| f \right\|_{\Phi_1, v}, \quad f \in \Omega; \tag{3.8}$$

$$\left\| \Re_a T^* (wg) \right\|_{\Psi_1, v} \le c_2 \left\| g u^{-1} \right\|_{\Psi_2, u}, \quad g \in M_+ ; \tag{3.9}$$

$$\left\| w\, T \Re_a^* (v f) \right\|_{\Phi_2, u} \le c_3 \left\| f \right\|_{\Phi_1, v}, \quad f \in M_+ . \tag{3.10}$$

Remark 3.5. Constants c_2, c_3 depend on $a \in (0,1)$, besides $0 < e(a) \le c_1 c_3^{-1} \le E(a) < \infty$; $0 < d \le c_2 c_3^{-1} \le D < \infty$;

where d, D do not depend of a .

Remark 3.6. Let us compare the inequalities (3.8) and (3.10).

$$\| wTf \|_{\Phi_2, u} \le c_1 \| f \|_{\Phi_1, v}, \quad f \in \Omega; \tag{3.8}$$

$$\left\| wT\mathfrak{R}_a^*(vf) \right\|_{\Phi_2, u} \le c_3 \| f \|_{\Phi_1, v}, \quad f \in M_+. \tag{3.10}$$

Their structures and right hand sides are the same, the only difference is that in the left side of (3.8) there is the Orlicz norm for function

$$w\,T\,f\,, \quad f \in \Omega,$$

and in the left side of (3.10) there is the norm of function

$$wT\mathfrak{R}_a^*(vf), \quad \mathfrak{R}_a^*(vf)(\tau) = \int_{\tau}^{\delta_a^{-1}(\tau)} \frac{f\,dV}{V}, \quad f \in M_+. \tag{3.11}$$

Thereby, in (3.10) the monotonicity condition is omitted for function f, thanks to the passage from $f \in \Omega$ to the integral average $\mathfrak{R}_a^*(vf)$.

Remark 3.7. Additionally, let the Young function Φ_1 in Theorem 3.4 satisfy Δ_2 - condition, that is

$$\exists C \in (1, \infty): \quad \Phi_1(2t) \le C\,\Phi_1(t), \quad t \in R_+ .$$

Then, according to Remark 3.3, the description remains true for $a = 0$, i. e., we have instead of $\Re_a^*(v\,f)$ in (3.11)

$$\Re_0^*(v\,f;\tau) = \int_\tau^\infty \frac{f(t)}{V(t)}\,dV, \ \tau \in R_+; \tag{3.12}$$

and (3.8) is equivalent now to the inequality

$$\left\| wT\left(\int_\tau^\infty \frac{f\,dV}{V} \right) \right\|_{\Phi_2, u} \le c_3 \| f \|_{\Phi_1, v}, \quad f \in M_+ . \tag{3.13}$$

4. REDUCTION THEOREM FOR MODULAR INEQUALITIES

Theorem 4.1. *In the conditions of Theorem 3.4 the following modular inequalities are equivalent: for* $f \in \Omega$

$$\Phi_2^{-1}\left\{ \int_{R_+} \Phi_2\left(wT(f)\right)u\,dt \right\} \le \Phi_1^{-1}\left\{ \int_{R_+} \Phi_1\left(c_1 f\right)v\,dt \right\}, \qquad (4.1)$$

or for $f \in M_+$

$$\Phi_2^{-1}\left\{ \int_{R_+} \Phi_2\left(wT\Re_a^*\left(v f\right)\right)u\,dt \right\} \le \Phi_1^{-1}\left\{ \int_{R_+} \Phi_1\left(c_3 f\right)v\,dt \right\} .$$

Moreover, c_3 *is connected with* c_1 *as in Theorem 3.4:*

$$0 < e\left(a\right) \le c_1\, c_3^{-1} \le E\left(a\right) < \infty .$$

Corollary 4.2. *In the conditions of Theorem 3.4 let us assume additionally that Young function* Φ_1 *satisfies* Δ_2*- condition. Then,*

$$\Phi_2^{-1}\left\{\int_{R_+}\Phi_2\left(wT\left(f\right)\right)u\,dt\right\}\le\Phi_1^{-1}\left\{\int_{R_+}\Phi_1\left(c_1 f\right)v\,dt\right\},$$

for $f\in\Omega$ *is equivalent to the following inequality: for* $f\in M_+$

$$\Phi_2^{-1}\left\{\int_{R_+}\Phi_2\left(wT\left(\int_\tau^\infty\frac{f\,dV}{V}\right)\right)u\,dt\right\}\le\Phi_1^{-1}\left\{\int_{R_+}\Phi_1\left(c_3 f\right)v\,dt\right\}.$$

5. APPLICATIONS FOR HARDY OPERATORS.

$$T\left(f;t\right)=\int_0^t f\left(\tau\right)d\tau,\quad t\in R_+. \qquad (5.1)$$

Here Φ_1, Φ_2 are N-functions: $\Phi_1\left(t\right)=\int_0^t \varphi_1\left(\tau\right)d\tau$, $\varphi_1\uparrow:\left(0,\infty\right)\to\left(0,\infty\right)$, $\varphi_1\left(0\right)=\varphi_1\left(+0\right)=0;\quad \varphi_1\left(+\infty\right)=\infty$.

$$\delta_b\left(t\right):=V^{-1}\left(b\,V\left(t\right)\right),\; b\in R_+,\quad t\in R_+.$$

$$k_1\left(t,\rho\right):=\frac{a}{V\left(\rho\right)}\begin{cases} t-\rho, & \rho<t\le\delta_{a^{-1}}\left(\rho\right),\\ \delta_{a^{-1}}\left(\rho\right)-\rho, & t>\delta_{a^{-1}}\left(\rho\right);\end{cases}$$

$$k_2\left(\rho,y\right)=\frac{a}{V\left(y\right)}\begin{cases} \delta_{a^{-1}}\left(y\right)-y, & 0<y\le\delta_a\left(\rho\right),\\ \rho-y, & \delta_a\left(\rho\right)<y<\rho;\end{cases}$$

Theorem 5.1. *In the conditions of Theorem 3.4 let* T *be Hardy operator (5.1),* Ψ_1, Ψ_2 *be complementary* N*-functions for* N*-functions* Φ_1, Φ_2 *;* $\Phi_2\left(\Phi_1^{-1}\right)$ *be convex and* $0<v<\infty$, $V(t)=\int_0^t v\,d\tau<\infty$, $t\in R_+$, $V(\infty)=\infty$. *Then,*

$$\Phi_2^{-1}\left\{\int_{R_+}\Phi_2(wT(f))u\,dt\right\}\le\Phi_1^{-1}\left\{\int_{R_+}\Phi_1(c_1 f)v\,dt\right\}, \quad (5.2)$$

for $f\in\Omega$ *is equivalent to:* $\exists C=C(c_1,a,)\in R_+$: *for all* $\varepsilon,\rho\in R_+$

$$\Phi_2^{-1}\left\{\int_\rho^\infty\Phi_2\left(\frac{w(\tau)}{C}\left\|\frac{k_2(\rho,\cdot)\chi_{(0,\rho)}(\cdot)}{\varepsilon}\right\|_{\Psi_1,\frac{\varepsilon v}{a}}\right)u(\tau)\,d\tau\right\}\le\Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

$$\Phi_2^{-1}\left\{\int_\rho^\infty\Phi_2\left(\frac{w(\tau)}{C}\left\|\frac{\chi_{(0,\rho)}(\cdot)}{\varepsilon}\right\|_{\Psi_1,\frac{\varepsilon v}{a}}k_1(\tau,\rho)\right)u(\tau)\,d\tau\right\}\le\Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

6. THE CASE OF WEIGHTED LEBESGUE SPACES

$$\Phi_1(t) = t^p / p, \quad \Phi_2(t) = t^q / q, \quad 1 < p \le q < \infty,$$

,

$$\Phi_1^{-1}(\tau) = (p\,\tau)^{1/p}, \quad \Phi_2^{-1}(\tau) = (q\,\tau)^{1/q};$$

(5.2) for Hardy operator : for $f \in \Omega$

$$\left\{ \int_{R_+} (wTf)^q\, u\, dt \right\}^{1/q} \le c_1 \left\{ \int_{R_+} f^p\, v\, dt \right\}^{1/p}, \qquad (5.3)$$

$$\tilde{k}_1(\tau, \rho) := \begin{cases} t - \rho, & \rho < t \le \delta_{a^{-1}}(\rho), \\ \delta_{a^{-1}}(\rho) - \rho, & t > \delta_{a^{-1}}(\rho); \end{cases} \qquad a \in (0,1);$$

$$\tilde{k}_2(\rho, y) := \frac{1}{V(y)} \begin{cases} \delta_{a^{-1}}(y) - y, & 0 < y \le \delta_a(\rho), \\ \rho - y, & \delta_a(\rho) < y < \rho. \end{cases}$$

$$\left\{ \int_{R_+} (wTf)^q u\,dt \right\}^{1/q} \le c_1 \left\{ \int_{R_+} f^p v\,dt \right\}^{1/p}, \qquad (5.3)$$

(5.3) is equivalent to: $\exists\, C = C(c_1, a) \in R_+$: for all $\rho \in R_+$

$$\left\{ \int_{(\rho,\infty)} w^q u\,dt \right\}^{1/q} \left\{ \int_{(0,\rho)} \tilde{k}_2(\rho, y)^{p'} v\,dy \right\}^{1/p'} \le C\, p^{1/p} (p')^{1/p'}$$

$$\frac{\left\{ \int_{(\rho,\infty)} w(\tau)^q \tilde{k}_1(\tau,\rho)^q u(\tau)\,d\tau \right\}^{1/q}}{V(\rho)^{1/p}} \le C\, p^{1/p} (p')^{1/p'}.$$

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Thanks for your attention!

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