Characterization of interpolation between Grand, small or classical Lebesgue spaces

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Let $(X_0, ||\cdot||_0)$, $(X_1, ||\cdot||_1)$ two Banach spaces contained continuously in a Hansdorff topological vector space (that is (X_0, X_1) is a compatible couple). For $g \in X_0 + X_1$, $t > 0$ one defines the so called K functional $K(g, t; X_0, X_1) \doteq K(g, t)$ by setting

$$\mathbf{K(g,t)} = \inf_{\mathbf{g=g_0+g_1}} \left(||\mathbf{g_0}||_\mathbf{0} + \mathbf{t}||\mathbf{g_1}||_\mathbf{1}\right).$$

For $0 \leqslant \theta \leqslant 1$, $1 \leqslant p \leqslant +\infty$, $\alpha \in \mathbb{R}$ we shall consider

$$(\mathbf{X_0}, \mathbf{X_1})_{\theta,\mathbf{p};\alpha} = \left\{\mathbf{g} \in \mathbf{X_0} + \mathbf{X_1},\ ||\mathbf{g}||_{\theta,\mathbf{p};\alpha} < +\infty\right\},$$

where

$$||\mathbf{g}||_{\theta,\mathbf{p};\alpha} := ||\mathbf{t}^{-\theta-\frac{1}{\mathbf{p}}}\left(\mathbf{1} - \operatorname{Log} \mathbf{t}\right)^{\alpha} \mathbf{K(g,t)}||_{\mathbf{L^p(0,1)}}.$$

Here $||\cdot||_V$ denotes the norm in a Banach space V.
The weighted Lebesgue space $L^p(0, 1; \omega)$, $0 < p \leqslant +\infty$ is endowed with the usual norm or quasi norm, where ω is a weight function on $(0, 1)$.

The grand Lebesgue space $L^{p),\alpha}(\Omega)$, with Ω a bounded (open) set of $\mathbb{R}^n$ whose measure is 1, $1 < p < +\infty, \alpha > 0$ defined as

$$L^{p),\alpha}(\Omega)=\left\{ f : \Omega \to \mathbb{R}\ \text{measurable}, ||f||_{p),\alpha} < +\infty \right\}.$$

where

$$|\mathbf{f}||_{\mathbf{p}),\alpha} := \sup_{\mathbf{0}<\varepsilon<\mathbf{p}-\mathbf{1}} \varepsilon^{\frac{\alpha}{\mathbf{p}-\varepsilon}} ||\mathbf{f}||_{\mathbf{p}-\varepsilon}.$$

The small Lebesgue space $L^{(p',\alpha}(\Omega)$ is defined as

$$\mathbf{L}^{(\mathbf{p}',\alpha}(\boldsymbol{\Omega}) = \|\left\{ \mathbf{f} : \boldsymbol{\Omega} \to \mathbb{R}, \mathbf{f} = \sum_{\mathbf{k}=\mathbf{1}}^{\infty} \mathbf{f}_{\mathbf{k}} \ (\text{convergence a.e.})\ ||\mathbf{f}||_{(\mathbf{p},\alpha} < +\infty \right\},$$

where

$$||\mathbf{f}||_{(\mathbf{p}',\alpha} := \inf_{\mathbf{f}=\sum_{\mathbf{k}=\mathbf{1}}^{\infty} \mathbf{f}_{\mathbf{k}}} \inf_{\mathbf{0}<\varepsilon<\mathbf{p}'-\mathbf{1}} \varepsilon^{-\frac{\alpha}{\mathbf{p}'-\varepsilon}} ||\mathbf{f}_{\mathbf{k}}||_{\mathbf{p}'-\varepsilon}.$$

$$\frac{1}{p'} + \frac{1}{p} = 1$$

The following characterizations of this spaces are known.

$$||\mathbf{f}||_{\mathbf{p}),\alpha} \approx \sup_{0<\mathbf{t}<1} (\mathbf{1} - \mathrm{Log}\,\mathbf{t})^{-\frac{\alpha}{\mathbf{p}}} \left(\int_{\mathbf{t}}^{\mathbf{1}} \mathbf{f}_*^{\mathbf{p}}(\sigma) \mathbf{d}\sigma \right)^{\frac{1}{\mathbf{p}}}.$$

$$||\mathbf{f}||_{(\mathbf{p},\alpha} \approx = \int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \mathrm{Log}\,\mathbf{t})^{-\frac{\alpha}{\mathbf{p}} + \alpha - \mathbf{1}} \left(\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{f}_*^{\mathbf{p}}(\sigma) \mathbf{d}\sigma \right)^{\frac{1}{\mathbf{p}}} \frac{\mathbf{dt}}{\mathbf{t}}.$$

Here, f_* is the decreasing rearrangement of $|f|$, say it is the generalized inverse of the distribution function

$$D_f(t) = measure\{x \in \Omega, |f(x)| > t\}, t \in \mathbb{R}_+.$$

Definition (Lorentz-Zygmund space)

For $1 \leqslant p,\ q \leqslant \infty,\ -\infty < \alpha < +\infty$, *the Lorentz-Zygmund space* $L^{p,q}(\mathrm{Log}\ \ L)^{\alpha}$ *consists of all functions* f *measurable such that* $||f||_{p,q;\alpha} < +\infty$

$$||\mathbf{f}||_{\mathbf{p},\mathbf{q};\alpha} = \begin{cases} \left(\int_0^1 \left[t^{\frac{1}{p}-\frac{1}{q}}(1-\mathrm{Log}\,t)^{\alpha} f_*(t)\right]^q dt\right)^{\frac{1}{q}} & \text{if } 1 \leqslant q < +\infty, \\ \sup_{0<t<1} t^{\frac{1}{p}}(1-\mathrm{Log}\,t)^{\alpha} f_*(t) & \text{if } q = +\infty \end{cases}$$

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Definition (Generalized Gamma space)

Let $1 \leqslant p,\ q \leqslant \infty$, the Generalized Gamma space with double weights $G\Gamma(p, m; w_1, w_2)$ consist of all functions such that

$$||\mathbf{f}||_{\mathbf{G(p,m;w_1,w_2)}} = \left[\int_0^1 \mathbf{w_1(t)} \left(\int_0^t \mathbf{f_*^p(\sigma)w_2(\sigma)d\sigma}\right)^{\frac{\mathbf{m}}{\mathbf{p}}} \mathbf{dt}\right]^{\frac{1}{\mathbf{m}}} < +\infty$$

with the obvious change for $m = +\infty$.

Theorem

Let $1 \leqslant p < q, \ \alpha > 0$. *Then*

$$L^{q),\alpha} = \left(L^p, L^q\right)_{1,\infty;-\frac{\alpha}{q}}.$$

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Computation of the K-functional for the couple $(L^{p),\alpha}, L^q)$

Theorem

Let $1 < p < q,\ \alpha > 0$. *Then*

$$K(f,t;L^{p),\alpha},L^q) \approx \sup_{0<s<\varphi(t)} (1-\operatorname{Log} s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(x)dx\right)^{\frac{1}{p}} + t\left(\int_{\varphi(t)}^1 f_*^q(s)ds\right)^{\frac{1}{q}}$$

where φ *is the inverse of the increasing function* $\psi(t) = t^{\frac{1}{p}-\frac{1}{q}}(1-\operatorname{Log} t)^{-\frac{\alpha}{p}}, \quad t\in(0,1)$. *Thus*

$$t = \varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1-\operatorname{Log}\varphi(t))^{-\frac{\alpha}{p}}.$$

Theorem

Let $0 < \theta < 1,\ 1 \leqslant r < +\infty$, $\alpha > 0$, $1 < p < q$. *Then*

$$\left(\mathbf{L}^{\mathbf{p}),\alpha}, \mathbf{L}^{\mathbf{q}),\alpha}\right)_{\theta,\mathbf{r}} = \mathbf{L}^{\mathbf{p}_\theta,\mathbf{r}}(\mathrm{Log}\,\mathbf{L})^{-\frac{\alpha}{\mathbf{p}_\theta}}$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{q}$.

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with $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{q},\ \dfrac{1}{p_\theta} + \dfrac{1}{p'_\theta} = 1.$

Theorem

Let $1 < a < +\infty, \ \beta \in \mathbb{R}, \ \beta \neq 0, \ 1 < r < +\infty$. *Then the Lorentz-Zygmund space* $L^{a,r}(\mathrm{Log}\, L)^{\beta}$ *is an interpolation space in the sense of Peetre of two Grand Lebesgue spaces if* $\beta < 0$ *and of two small Lebesgue spaces if* $\beta > 0$

Theorem

Let $1 < p < q < +\infty$. *Then for all* $t > 0$, $f \in L^{(p}$

$$K(f, t; L^{(p}, L^{q)}) \approx \int_0^{\varphi(t)} (1 - \operatorname{Log} s)^{\frac{-1}{p}} \left(\int_0^s f_*^p(\tau) d\tau \right)^{\frac{1}{p}} \frac{ds}{s}$$

$$+ \quad (1 - \operatorname{Log} t)^{\frac{p-1}{p}} \left(\int_0^{\varphi(t)} f_*^p(\tau) d\tau \right)^{\frac{1}{p}}$$

$$+ t \sup_{\frac{1}{2}\varphi(t) < s < 1} (1 - \operatorname{Log} s)^{\frac{-1}{q}} \left(\int_s^1 f_*^q(\tau) d\tau \right)^{\frac{1}{q}}$$

where φ *is an invertible function from* $[0, 1]$ *into itself satisfying the equivalence*

$$\varphi(t)^{\frac{1}{p} - \frac{1}{q}} (1 - \operatorname{Log} \varphi(t))^{\frac{p-q+pq}{pq}} \approx t.$$

Theorem

Let $0 < \theta < 1,\ 1 \leqslant r < +\infty,\ p < q$. *Then*

$$\left(\mathbf{L}^{(\mathbf{p}}, \mathbf{L}^{\mathbf{q})}\right)_{\theta,\mathbf{r}} = \mathbf{L}^{\mathbf{p}_\theta,\mathbf{r}} \left(\mathrm{Log}\,\mathbf{L}\right)^{\alpha_\theta}$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{q}, \quad \alpha_\theta = 1 - \theta - \dfrac{1}{p_\theta}.$

K-function for $(L^{p)}, L^{(p})$

Theorem

For $1 < p < +\infty,\ 0 < t < 1,\ f \in L^{p)} + L^{(p}$, *one has*

$$K(f, t; L^{p)}, L^{(p}) \approx \sup_{0<s<e^{1-\frac{1}{t}}} (1 - \operatorname{Log} s)^{\frac{-1}{p}} \left(\int_s^{e^{1-\frac{1}{t}}} f_*^p(x)dx \right)^{\frac{1}{p}}$$

$$+ t \int_{e^{1-\frac{1}{t}}}^1 (1 - \operatorname{Log} s)^{\frac{-1}{p}} \left(\int_{e^{1-\frac{1}{t}}}^s f_*^p(x)dx \right)^{\frac{1}{p}} \frac{ds}{s}.$$

Theorem

Let $1 < p < +\infty,\ 0 < \theta < 1,\ 1 \leqslant r < +\infty.$ *Then* $Z_{\theta,r} \doteq (L^{p)}, L^{(p})_{\theta,r}$ *has the following equivalent norm:*

For $f \in Z_{\theta,r},\ \beta_\theta = \theta - \dfrac{1}{p} - \dfrac{1}{r}$

$$||f||_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta_\theta} \left(\int_t^1 f_*^p(s)ds\right)^{\frac{1}{p}}\right]^r \frac{dt}{t}\right]^{\frac{1}{r}}, \qquad \theta < \frac{1}{p}$$

$$||f||_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta_\theta} \left(\int_0^t f_*^p(s)ds\right)^{\frac{1}{p}}\right]^r \frac{dt}{t}\right]^{\frac{1}{p}}, \qquad \theta > \frac{1}{p}$$

$$||f||_{Z_{\theta,r}} \approx \left[\sum_{k=0}^{+\infty} \left(\int_{2^{1-2^{k+1}}}^{2^{1-2^k}} f_*^p(s)ds\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}, \qquad \theta = \frac{1}{p}$$

In particular

$$(\mathbf{L^{p)}}, \mathbf{L^{(p}})_{\frac{1}{p},\mathbf{p}} = \mathbf{L^p} \qquad \textit{and} \qquad (\mathbf{L^{p)}}, \mathbf{L^{(p}})_{\theta,\mathbf{p}} = \mathbf{L^{p,p}}(\mathrm{Log}\,\mathbf{L})^{\theta - \frac{1}{\mathbf{p}}}.$$

Corollary

$\theta \in]0,1[$ and $f \in Z_{\theta,r}$

$$||\mathbf{f}||^{\mathbf{r}}_{\mathbf{Z}_{\theta,\mathbf{r}}} \approx \sum_{\mathbf{k}\in\mathbb{N}} \mathbf{2}^{\mathbf{kr}(\theta-\frac{\mathbf{1}}{\mathbf{p}})} \left(\int_{\mathbf{t}_{\mathbf{k}+1}}^{\mathbf{t}_{\mathbf{k}}} \mathbf{f}^{\mathbf{p}}_{*}(\mathbf{y})\mathbf{dy}\right)^{\frac{\mathbf{r}}{\mathbf{p}}},$$

where $t_k = 2^{1-2^k}, \ k \in \mathbb{N}$.

Theorem

Let $1 < p < +\infty, \ 0 < \theta < 1, \ 1 \leqslant r < +\infty$. Then

$$(\mathbf{L}^{\mathbf{p})}, \mathbf{L}^{(\mathbf{p}})_{\theta,\mathbf{r}} = \mathbf{G\Gamma}(\mathbf{p},\mathbf{r};\mathbf{w}_1,\mathbf{w}_2),$$

with $w_1(t) = t^{-1}(1-\operatorname{Log} t)^{\theta r-1}, \ w_2(t) = (1-\operatorname{Log} t)^{-1}, \ t \in (0,1)$.
i.e.

$$||\mathbf{f}||_{\mathbf{Z}_{\theta,\mathbf{r}}} \approx \left[\int_0^1 (\mathbf{1}-\operatorname{Log}\mathbf{t})^{\theta\mathbf{r}} \left(\int_0^{\mathbf{t}} (\mathbf{1}-\operatorname{Log}\mathbf{x})^{-1}\mathbf{f}^{\mathbf{p}}_{*}(\mathbf{x})\mathbf{dx}\right)^{\frac{\mathbf{r}}{\mathbf{p}}} \frac{\mathbf{dt}}{(\mathbf{1}-\operatorname{Log}\mathbf{t})\mathbf{t}}\right]^{\frac{\mathbf{1}}{\mathbf{r}}}.$$