Homogeneous Triebel-Lizorkin spaces associated with operators

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Outline of Topics

1. The general setting (M, ρ, μ, L)
2. Distributions associated with operators
3. Generalized polynomials associated with operators
4. Homogeneous Triebel-Lizorkin spaces
5. Final remarks

Doubling spaces associated with operators

Let (M, ρ) be a metric space and μ a positive measure such that

1. (Doubling volume property) There exists a constant $c_0 > 1$:

$$|B(x, 2r)| \leq c_0 |B(x, r)|, \ \forall x \in M, \ r > 0. \tag{1}$$

   The role of the dimension plays:

$$d := \log_2 c_0 > 0. \tag{2}$$

2. (Operator) There exists a self-adjoint non-negative operator L mapping real-valued functions to real-valued functions whose heat kernel is Markov and satisfies the upper bounds:

$$|p_t(x,y)| \le \frac{c_1 \exp\left(-\frac{c_2\rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}, \ \forall x,y \in M, \ \ t>0. \qquad (3)$$

and the Hölder continuity

$$|p_t(x,y) - p_t(x,y')| \le c_1\left(\frac{\rho(y,y')}{\sqrt{t}}\right)^{\alpha} \frac{\exp\left(-\frac{c_2\rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}, \quad (4)$$

for some $\alpha > 0$ and $\forall x, y \in M$ such that $\rho(y,y') \le \sqrt{t}$ and $t > 0$.

Examples

This very general setting covers e.g.

1. $\mathbb{R}^n$ associated with uniformly elliptic divergence form operator,
2. Riemannian manifolds with non-negative Ricci curvature, associated with the Laplace-Beltrami operator,
3. Lie groups of polynomial volume growth with sub-laplacians,
4. the ball,
5. the sphere,
6. the interval with the Jacobi operator...

Test functions: Schwartz class on $\mathbb{R}^n$

We say that the function $\phi : \mathbb{R}^n \to \mathbb{R}$ is in the Schwartz class $\phi \in \mathcal{S}(\mathbb{R}^n)$ when:

(α) $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and

(β) For every $\gamma \in \mathbb{N}_0^n$, $k \in \mathbb{N}$, there exists $c = c_{\gamma,k} > 0$ such that

$$|\partial^\gamma \phi(x)| \leq c(1 + |x|)^{-k}, \ \forall\, x \in \mathbb{R}^n. \tag{5}$$

Tempered distributions on $\mathbb{R}^n$

The dual space $\mathcal{S}'(\mathbb{R}^n)$ will be the class of *tempered distributions*. We represent the action of $f \in \mathcal{S}'(\mathbb{R}^n)$ on $\phi \in \mathcal{S}$ by

$$f(\phi) = \langle f, \phi \rangle, \ \forall \ \phi \in \mathcal{S}. \tag{6}$$

- We can identify proper functions f with the distribution L_f

$$L_f(\phi) := \int_{\mathbb{R}^n} f(x)\phi(x)dx, \ \ \phi \in \mathcal{S}. \tag{7}$$

For example the following functions can be viewed as distributions:

1. Functions $f \in L^p, \ p \in [1, \infty]$.
2. The set $\mathcal{P}$ of Polynomials: $\mathcal{P} \subset \mathcal{S}'$

Distributions associated with operators

We say that $\phi : M \to \mathbb{R}$ is a test function $\phi \in \mathcal{S} = \mathcal{S}(L)$ when:
(α) $\phi \in \cap_{m \geq 0} Dom(L^m)$
(β) For every $m, \ell \geq 0$ there exists $c = c_{m,\ell} > 0$ such that

$$|L^m \phi(x)| \leq c\big(1 + \rho(x, x_0)\big)^{-\ell}, \quad \forall x \in M. \tag{8}$$

Here $x_0 \in M$ is selected arbitrarily and fixed for now on and its particular selection is unimportant.

- Operator L replaces derivatives.
- The space $\mathcal{S}' = \mathcal{S}'(L)$ of distributions on M is defined as the set of all continuous linear functionals on $\mathcal{S}$.
- On $\mathbb{R}^n$, torus, sphere, ball with the Laplacian, $\mathcal{S}(L)$, $\mathcal{S}'(L)$ coincide with the Schwartz class and tempered distributions respectively.

Papers

1. G. Kerkyacharian and P. Petrushev. Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces. Trans. Amer. Math. Soc., 367(1): 121–189, 2015.
2. S. Dekel, G. Kerkyacharian, G. Kyriazis, and P. Petrushev. Compactly supported frames for spaces of distributions associated with nonnegative self-adjoint operators. Stud. Math., 225(2): 115–163, 2014.
3. L. Liu, D. Yang, and W. Yuan. Besov-type and Triebel-Lizorkin-type spaces associated with heat kernels. Collect. Math., 67(2): 247-310, 2016.
4. S. Dekel, G. Kerkyacharian, G. Kyriazis, and P. Petrushev. Hardy spaces associated with non-negative self-adjoint operators. Stud. Math., 239 (1): 17-54, 2017.

Papers with my contribution

1. G., M. Nielsen. Pseudodifferential operators on spaces of distributions associated with non-negative self-adjoint operators. J. Fourier Anal. Appl. 23 (2017), no. 2, 344-378.
2. G., G. Kerkyacharian, G. Kyriazis, and P. Petrushev. Homogeneous spaces of distributions associated with non-negative self-adjoint operators. Jour. Math. Anal. Appl. 449 (2017) no. 2, 1382-1412.
3. G., M. Nielsen. Spectral multipliers on spaces of distributions associated with non-negative self-adjoint operators. preprint.
4. G., G. Kerkyacharian, G. Kyriazis, and P. Petrushev. Atomic and molecular decomposition of spaces of distributions associated with non-negative self-adjoint operators. preprint.

Generalized polynomials associated with operators

Definition

We define the set $\mathbf{P}_m = \mathbf{P}_m(L)$ of generalized polynomials of degree m ($m \geq 0$) by

$$\mathbf{P}_m := \{g \in \mathcal{S}' : L^m g = 0\} \tag{9}$$

and set $\mathbf{P} := \mathbf{P}(L) := \cup_{m \geq 0} \mathbf{P}_m$ the class of generalized polynomials.

- $g \in \mathbf{P}_m \Leftrightarrow \langle g, L^m \phi \rangle = 0$ for all $\phi \in \mathcal{S}$.
- We define an *equivalence* $f \sim g$ on $\mathcal{S}'$ by

$$f \sim g \Longleftrightarrow f - g \in \mathbf{P}.$$

We denote by $\mathcal{S}'/\mathbf{P}$ the set of all equivalent classes in $\mathcal{S}'$.

- Operator L replaces derivatives.

Homogeneous Triebel-Lizorkin spaces on $\mathbb{R}^n$

(α) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \varphi \subset \{2^{-1} \leq |\xi| \leq 2\}, \quad |\varphi(\xi)| \geq c, \quad \text{for } 3/5 \leq |\xi| \leq 5/3. \tag{10}$$

(β) We set the dilations

$$\varphi_j(\xi) := \varphi(2^{-j}\xi), \quad \forall j \in \mathbb{N} \tag{11}$$

Homogeneous Triebel-Lizorkin spaces on $\mathbb{R}^n$

(γ) We define the operator action $\varphi_j(D)g$, for $g \in \mathcal{S}(\mathbb{R}^n)$

$$\varphi_j(D)g(x) := \mathcal{F}^{-1}(\varphi_j) * g(x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\varphi_j)(x-y)g(y)dy \quad (12)$$

and extend to $\varphi_j(D)f$, $f \in \mathcal{S}'(\mathbb{R}^n)$ by duality.

- Note that $\varphi_j(D)$ is and integral operator on $\mathcal{S}(\mathbb{R}^n)$ with kernel

$$|\mathcal{F}^{-1}(\varphi_j)(x-y)| = |2^{jn}(\mathcal{F}^{-1}\varphi)(2^j(x-y))| \leq c2^{jn}(1+2^j|x-y|)^{-k}. \quad (13)$$

- For every $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the action $\phi(D)f$ is a function with at most polynomial growth

$$|\phi(D)f(x)| \leq c(1+|x|)^m. \quad (14)$$

Homogeneous Triebel-Lizorkin spaces on $\mathbb{R}^n$

Definition

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $f \in \mathcal{S}'/\mathcal{P}$. We say that f belongs to Homogeneous Triebel-Lizorkin space $\dot{F}^s_{pq}$, when

$$\|f\|_{\dot{F}^s_{pq}} := \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^{sj} |\varphi_j(D) f|\right)^q \right)^{1/q} \right\|_p < \infty. \qquad (15)$$

Admissible functions

(α) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that

$$\operatorname{supp} \varphi \subset [2^{-1}, 2], \ |\varphi(\lambda)| \geq c, \ \lambda \in [2^{-3/4}, 2^{3/4}], \tag{16}$$

(β) We set the dilations

$$\varphi_j(\lambda) = \varphi(2^{-j}\lambda), \ \text{ for every } j \in \mathbb{Z}. \tag{17}$$

Operators (Spectral multipliers)

(γ) Then the spectral multipliers $\varphi_j(\sqrt{L})$

$$\varphi_j(\sqrt{L}) := \int_0^\infty \varphi_j(\sqrt{\lambda})dE_\lambda, \quad dE_\lambda \text{ the spectral measure of } L, \tag{18}$$

are integral operators with kernels decaying as

$$|\varphi_j(\sqrt{L})(x,y)| \le c\frac{(1+2^j\rho(x,y))^{-k}}{(|B(x,2^{-j})||B(y,2^{-j})|)^{1/2}}. \tag{19}$$

- For every $g \in \mathcal{S}(L)$

$$\varphi_j(\sqrt{L})g(x) = \int_M \varphi_j(\sqrt{L})(x,y)g(y)dy.$$

Extend to $f \in \mathcal{S}'(L)$ by duality.

- For every $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ compactly supported away from 0, the action $\phi(\sqrt{L})f$ is a continuous function with

$$|\phi(\sqrt{L})f(x)| \le c(1+\rho(x,x_0))^m. \tag{20}$$

Operators (Spectral multipliers)

The operator L replaces the Fourier transform.

Triebel-Lizorkin spaces on (M, ρ, μ, L)

Let φ admissible.

Definition

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. An $f \in \mathcal{S}'/\mathbf{P}$ belongs to
(α) Homogeneous Triebel-Lizorkin space $\dot{F}^s_{pq} = \dot{F}^s_{pq}(L)$ if

$$\|f\|_{\dot{F}^s_{pq}} := \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(2^{js} |\varphi_j(\sqrt{L}) f(\cdot)| \Big)^q \Big)^{1/q} \Big\|_{L^p} < \infty. \tag{21}$$

(β) Non-classical Homogeneous Triebel-Lizorkin space
$\dot{\tilde{F}}^s_{pq} = \dot{\tilde{F}}^s_{pq}(L)$ if

$$\|f\|_{\dot{\tilde{F}}^s_{pq}} := \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(|B(\cdot, 2^{-j})|^{-s/d} |\varphi_j(\sqrt{L}) f(\cdot)| \Big)^q \Big)^{1/q} \Big\|_{L^p} < \infty. \tag{22}$$

Results

1. Discrete decomposition: Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Then

$$\|f\|_{\dot{F}^s_{pq}} \sim \|\langle f, \psi_\xi \rangle\|_{\dot{f}^s_{pq}}.$$

2. Let $s \in \mathbb{R}$, $m := \lceil s \rceil$, $1 < p < \infty$ and $1 < q \le \infty$. Then

$$\|f\|_{\dot{F}^s_{pq}} \sim \|f\|_{\dot{F}^s_{pq}(H)} := \left\| \left(\int_0^\infty \left(t^{-s/2} |(tL)^{m/2} e^{-tL} f(\cdot)| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_p.$$

3. Let $s > 0$, $1 \le p \le \infty$ and $0 < q \le \infty$. Then

$$B^s_{pq} = L^p \cap \dot{B}^s_{pq} \text{ and } F^s_{pq} = L^p \cap \dot{F}^s_{pq}.$$

4. For every $s \in \mathbb{R}$ and $1 < p < \infty$,

$$\|f\|_{\dot{F}^s_{p2}} \sim \|f\|_{\dot{L}^p_s} := \|L^{s/2} f\|_{L^p}$$

Open problems

1. General setting: Embeddings, Duals, interpolation, $p = \infty$, non smooth atomic decomposition, sharp spectral multiplier results and composition of symbols.
2. Morrey and Morrey-Campanato spaces
3. Homogeneous T-L type spaces and Q_p spaces.
4. Generalization of FIO.

Thanks for your attention!