Compactness Results for a Class of Limiting Interpolation Methods

New perspectives in the theory of function spaces and their applications

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Compactness Results for a Class of Limiting Interpolation Methods

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Outline

1. A Problem
2. The real interpolation method
3. Compactness Theorems
4. Limit interpolation methods
5. Slowly varying functions and symmetric spaces
6. Compactness again

A Compactness Theorem

Theorem (Meskhi, PEMS 2001)

Let $T : L_q(0,1) \longrightarrow L_q(0,1)$ *be the integral operator given by*

$$Tf(x) = \int_0^1 K(x,y)f(y)dy$$

A Compactness Theorem

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Let $T : L_q(0,1) \longrightarrow L_q(0,1)$ *be the integral operator given by*

$$Tf(x) = \int_0^1 K(x,y)f(y)dy$$

and assume that

$$\Big\| \|K(x,\cdot)\|_{L_{q'}(0,1)} \Big\|_{L_q(0,1)} \quad \text{is finite for some } 1 < q < \infty. \tag{1.1}$$

Then

$$T : L_q(0,1) \longrightarrow L_q(0,1)$$

is compact

Grand and small Lebesgue spaces.

Let $\Omega \subset \mathbb{R}^n$ measurable set of finite Lebesgue measure $|\Omega| < \infty$.

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Definition

For $1 < p < \infty$

$$L^{p)}(\Omega) = \left\{ f \in \mathcal{M}_0, \|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \right)^{\frac{1}{p-\varepsilon}} \|f\|_{L_{p-\varepsilon}} \right\}$$

Grand and small Lebesgue spaces.

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$$L^p(\Omega) \subset L^{p)}(\Omega) \subset L^{p-\varepsilon}(\Omega)$$

for $0 < \varepsilon < p - 1$

Definition

For $1 < p < \infty$ *we consider those functions* f *which can be represented as*

$$f = \sum_{k=1}^{\infty} f_k, \qquad f_k \in \mathcal{M}_0$$

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for which

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \frac{1}{\varepsilon} \left(\frac{\varepsilon}{|\Omega|} \right)^{\frac{1}{(p-\varepsilon)'}} \|f_k\|_{L_{(p-\varepsilon)'}} < \infty$$

Small Lebesgue Space

Definition

For $1 < p < \infty$ we consider those functions f which can be represented as

$$f = \sum_{k=1}^{\infty} f_k, \qquad f_k \in \mathcal{M}_0$$

$$\|f\|_{(p'} = \inf_{f=\sum_{k=1}^{\infty} f_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \frac{1}{\varepsilon} \left(\frac{\varepsilon}{|\Omega|} \right)^{\frac{1}{(p-\varepsilon)'}} \|f_k\|_{L_{(p-\varepsilon)'}} \right\}$$

Small Lebesgue Space

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For all $\varepsilon > 0$

$$L^{p'+\varepsilon} \subset L^{(p'} \subset L^{p'}$$

Question

$$T : L_{q)}(0,1) \longrightarrow L_{q)}(0,1)$$

$$T : L_{(q}(0,1) \longrightarrow L_{(q}(0,1)$$

Question

Are

$$T : L_{q)}(0,1) \longrightarrow L_{q)}(0,1)$$

$$T : L_{(q}(0,1) \longrightarrow L_{(q}(0,1)$$

compact???

The Real Interpolation Method

If $A_0, A_1 \hookrightarrow \mathcal{U}$ we say that

$$\overline{A} = (A_0, A_1) \quad \text{is an interpolation couple.}$$

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1$$

The Real Interpolation Method

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1$$

$$K(t, a; \overline{A}) = \inf_{a=a_0+a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\}$$

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1$$

$$K(t, a; \overline{A}) = \inf_{a = a_0 + a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\}$$

Definition

$0 < \theta < 1$ *and* $0 < q \leq \infty$. *The space* $\overline{A}^K_{\theta,q}$ *consists of all those elements s.t.*

$$\|a\|^K_{\theta,q} = \|t^{-\theta} K(t, a)\|_{L_q(\frac{dt}{t})} < \infty$$

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

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The interpolation property

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$$\begin{aligned} T :& A_0 \longrightarrow B_0 \\ T :& A_1 \longrightarrow B_1 \end{aligned}$$

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$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

$$\begin{aligned} T :& A_0 \longrightarrow B_0 \\ T :& A_1 \longrightarrow B_1 \end{aligned} \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q}$$

The interpolation property

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

$$\begin{matrix} T : A_0 \longrightarrow B_0 \\ T : A_1 \longrightarrow B_1 \end{matrix} \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q}$$

$$\| T \|_{\overline{A}^K_{\theta,q}, \overline{B}^K_{\theta,q}} \leq C \| T \|^{1-\theta}_{A_0,B_0} \| T \|^{\theta}_{A_1,B_1}$$

Examples

Let $0 < p_0 < p_1 \leq \infty$.
Then, for $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

$$\left(L_{p_0}, L_{p_1}\right)_{\theta,p} = L_p$$

Theorem

Let $0 < p_0 \neq p_1 \leq \infty$ *and* $0 < q_0 \neq q_1 \leq \infty$, *and let*

$$T : L_{p_0} \longrightarrow L_{q_0}$$
$$T : L_{p_1} \longrightarrow L_{q_1}$$

then, if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *for some* $0 < \theta < 1$ *and* $p \leq q$

$$T : L_p \longrightarrow L_q$$

$$\|T\|_{L_p, L_q} \leq C \|T\|^{1-\theta}_{L_{p_0}, L_{q_0}} \|T\|^{\theta}_{L_{p_1}, L_{q_1}}$$

Krasnosel'skii theorem

Theorem

Let $0 < p_0 \neq p_1 \leq \infty$ *and* $0 < q_0 \neq q_1 \leq \infty$ *with* $q_0 < \infty$, *and let*

$$T : L_{p_0} \longrightarrow L_{q_0} \quad \text{compact}$$
$$T : L_{p_1} \longrightarrow L_{q_1}$$

then, if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *for some* $0 < \theta < 1$

$$T : L_p \longrightarrow L_q \quad \text{is compact}$$

One-sided compactness theorem

Let $0 < \theta < 1$ and $1 \leq p \leq \infty$,

$$T : A_0 \longrightarrow B_0 \text{ compact}$$
$$T : A_1 \longrightarrow B_1$$

One-sided compactness theorem

Let $0 < \theta < 1$ and $1 \leq p \leq \infty$,

$$\left.\begin{array}{l} T : A_0 \longrightarrow B_0 \ \text{ compact} \\ T : A_1 \longrightarrow B_1 \end{array}\right. \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q} \ \text{ compact}$$

One-sided compactness theorem

Theorem Cobos, Kühn & Schonbek, JFA 1992

Let $0 < \theta < 1$ *and* $1 \leq p \leq \infty$,

$$\left.\begin{array}{l} T : A_0 \longrightarrow B_0 \;\; \textit{compact} \\ T : A_1 \longrightarrow B_1 \end{array}\right. \Rightarrow T : \overline{A}^K_{\theta,q} \longrightarrow \overline{B}^K_{\theta,q} \;\; \textit{compact}$$

Classic real interpolation scale

Let $0 < p_0 < p_1 \leq \infty$.
Then, for $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

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For $1 \leq q \leq \infty$

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Then, for $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

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For $1 \leq q \leq \infty$

$$\left(L_{p_0}, L_{p_1}\right)_{\theta,q} = L_{p,q}$$

But either $L_{(p}$ nor $L_{p)}$ belong to the scale $\left(L_{p_0}, L_{p_1}\right)_{\theta,q}$.

The problem of identifying limit spaces of the real interpolation scale has been studied by several authors:

Cobos, F., Fernández-Cabrera, L. M., Kühn T. and Ullrich, T., On an extreme class of real interpolation spaces, *J. Funct. Anal.* **256** (2009), 2321-2366.

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Slowly varying functions and symmetric spaces

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,q} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta} K(t,a)\|_{L_q(\frac{dt}{t})} < \infty$$

Slowly varying functions and symmetric spaces

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,\mathbf{b},q} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta}\mathbf{b(t)}K(t,a)\|_{L_q(\frac{dt}{t})} < \infty$$

$$A_0 \cap A_1 \hookrightarrow \overline{A}^K_{\theta,\mathbf{b},\mathbf{E}} \hookrightarrow A_0 + A_1$$

$$\|a\|^K_{\theta,q} = \|t^{-\theta}\mathbf{b}(\mathbf{t})K(t,a)\|_{\widetilde{\mathbf{E}}} < \infty$$

for $0 \leq \theta \leq 1$.

When the functions b varies very slowly we need some modifications in our methods.

$$\|a\|_{0,b,E} = \|b(t)K(t,a)\|_{\widehat{E}}$$

where $\widehat{E}$ corresponds to E but with respect to the measure $\nu(A) = \int \chi_A \frac{dt}{tL(t)}$

L is determined by b.

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Theorem

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *be Banach couples and let* $T \in \mathcal{L}(\bar{A}, \bar{B})$. *Let*

$$T : A_0 \longrightarrow B_0$$
$$T : A_1 \longrightarrow B_1 \quad \textit{compact}$$

Then, for any rearrangement invariant function space E, $\varphi \in \mathcal{P}$ *and* $0 < \pi_{\Phi_1}$ *or* $\rho_{\Phi_1} < 0$, *the operator*

$$T : (A_0, A_1)^{K}_{1,\varphi,\widehat{E}} \longrightarrow (B_0, B_1)^{K}_{1,\varphi,\widehat{E}}$$

is compact.

Question

$$T : L_{q)}(0,1) \longrightarrow L_{q)}(0,1)$$

$$T : L_{(q}(0,1) \longrightarrow L_{(q}(0,1)$$

Question

Are

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Answer

Theorem

Let $T : L_q(0,1) \longrightarrow L_q(0,1)$ *be the integral operator given by*

$$Tf(x) = \int_0^1 K(x,y)f(y)dy$$

and assume that $\left\| \|K(x,\cdot)\|_{L_{q'}(0,1)} \right\|_{L_q(0,1)} < \infty$ *for some* $1 < q < \infty$. *Then,*

Answer

Theorem

$T : L_q(0,1) \longrightarrow L_q(0,1)$ where $Tf(x) = \int_0^1 K(x,y)f(y)dy$ with

$$\Big\| \|K(x,\cdot)\|_{L_{q'}(0,1)} \Big\|_{L_q(0,1)} < \infty$$

a) *if in addition $T : L_1(0,1) \longrightarrow L_1(0,1)$ is bounded, the restriction $T : L^{q)} \longrightarrow L^{q)}$ is compact.*

Answer

Theorem

$T : L_q(0,1) \longrightarrow L_q(0,1)$ *where* $Tf(x) = \int_0^1 K(x,y)f(y)dy$ *with*

$$\Big\| \|K(x,\cdot)\|_{L_{q'}(0,1)} \Big\|_{L_q(0,1)} < \infty$$

a) *if in addition* $T : L_1(0,1) \longrightarrow L_1(0,1)$ *is bounded, the restriction* $T : L^{q)} \longrightarrow L^{q)}$ *is compact.*

b) *if in addition* $T : L_\infty(0,1) \longrightarrow L_\infty(0,1)$ *is bounded, the restriction* $T : L^{(q} \longrightarrow L^{(q}$ *is compact.*

Proof

Fiorenza and Karadzhov Z. Anal. Anwend. (2003)

$$L^{q)} = (L_1, L_q)_{1,\ell^{-1/q},\widehat{L}_\infty}$$
$$L^{(q} = (L_q, L_\infty)_{0,\ell^{1/q'},\widehat{L}_1}$$

Proof

Fiorenza and Karadzhov Z. Anal. Anwend. (2003)

$$L^{q)} = (L_1, L_q)_{1,\ell^{-1/q},\widehat{L}_\infty}$$
$$L^{(q} = (L_q, L_\infty)_{0,\ell^{1/q'},\widehat{L}_1}$$

Case a)

$$\begin{array}{l} T : L_1 \longrightarrow L_1 \\ T : L_q \longrightarrow L_q \end{array} \Longrightarrow T : L_{q)} \longrightarrow L_{q)}$$

Proof

Fiorenza and Karadzhov Z. Anal. Anwend. (2003)

$$L^{q)} = (L_1, L_q)_{1,\ell^{-1/q},\widehat{L}_\infty}$$
$$L^{(q} = (L_q, L_\infty)_{0,\ell^{1/q'},\widehat{L}_1}$$

$$\begin{matrix} T : L_1 \longrightarrow L_1 \\ T : L_q \underset{\text{Compact}}{\longrightarrow} L_q \end{matrix} \Longrightarrow T : L_{q)} \longrightarrow L_{q)} \text{ Compact}$$

■

Example

$K_q(x, y) : (0, 1) \times (0, 1) \longrightarrow (0, \infty)$

$$K_q(x, y) = \frac{v(x)}{x^{1/q+\alpha}} \chi_{(0,x)}(y) y^\beta$$

where v is an slowly varying function, $1 < q < \infty$ and $\beta > \alpha > 0$. Then

$$T_q : L^{q)} \longrightarrow L^{q)}$$
$$T_q : L^{(q} \longrightarrow L^{(q}$$

are compact operators.

Example

$$T_q : L_1(0,1) \longrightarrow L_1(0,1) \text{ bounded.}$$

Example

$$T_q : L_1(0,1) \longrightarrow L_1(0,1) \text{ bounded.}$$

$$\begin{aligned} \|Tf\|_{L_1} &= \int_0^1 \int_0^t \frac{v(t)}{t^{1/q+\alpha}} s^\beta f(s)\,ds\,dt \\ &\leq \|f\|_{L_1} \int_0^1 \frac{v(t)}{t^{1/q+\alpha}} t^\beta dt \\ &\sim \|f\|_{L_1} \int_0^1 v(t) t^{\beta-\alpha+1-1/q} \frac{dt}{t} < \infty. \end{aligned}$$

Example

$$T_q : L_\infty(0,1) \longrightarrow L_\infty(0,1) \text{ bounded.}$$

Example

$$T_q : L_\infty(0,1) \longrightarrow L_\infty(0,1) \text{ bounded.}$$

$$\begin{aligned} \|Tf\|_{L_\infty} &= \sup_{t>0} \int_0^t \frac{v(t)}{t^{1/q+\alpha}} s^\beta f(s)ds \\ &\lesssim \|f\|_{L_\infty} \sup_{t>0} v(t) t^{\beta+1-\alpha-1/q} < \infty. \end{aligned}$$

Example

Moreover

$$\left\| \left\| K_q(x, \cdot) \right\|_{L_{q'}(0,1)} \right\|_{L_q(0,1)} < \infty$$

Example

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$$\left\| \left\| K_q(x,\cdot) \right\|_{L_{q'}(0,1)} \right\|_{L_q(0,1)} < \infty$$

$$\begin{aligned}
&\left\| \left\| \frac{v(t)}{t^{1/q+\alpha}} \chi_{(0,t)}(s) s^{\beta} \right\|_{L_{q'}(0,1)(ds)} \right\|_{L_q(0,1)(dt)} \\
&\qquad = \left(\int_0^1 \left(\int_0^t \left[\frac{v(t)}{t^{1/q+\alpha}} \right]^{q'} s^{\beta q'} ds \right)^{q/q'} dt \right)^{1/q} \\
&\qquad \sim \left(\int_0^1 \frac{v(t)^q}{t^{1+\alpha q}} t^{\beta q + q/q'} dt \right)^{1/q} \\
&\qquad = \left(\int_0^1 v(t)^q t^{q(\beta-\alpha+1/q')} \frac{dt}{t} \right)^{1/q} < \infty.
\end{aligned}$$

Example

Then

1. $T_q : L_1(0,1) \longrightarrow L_1(0,1)$

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2. $T_q : L_\infty(0,1) \longrightarrow L_\infty(0,1)$

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1. $T_q : L_1(0,1) \longrightarrow L_1(0,1)$
2. $T_q : L_\infty(0,1) \longrightarrow L_\infty(0,1)$
3. $\left\| \left\| K_q(x,\cdot) \right\|_{L_{q'}(0,1)} \right\|_{L_q(0,1)} < \infty$

Example

Then

1. $T_q : L_1(0,1) \longrightarrow L_1(0,1)$
2. $T_q : L_\infty(0,1) \longrightarrow L_\infty(0,1)$
3. $\left\| \left\| K_q(x,\cdot) \right\|_{L_{q'}(0,1)} \right\|_{L_q(0,1)} < \infty$

Therefore,

$$T_q : L^{q)} \longrightarrow L^{q)}$$
$$T_q : L^{(q} \longrightarrow L^{(q}$$

are compact operators.

THANK YOU
FOR
YOUR ATTENTION.