Function Spaces and Robustness Considerations

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Motivation

This talk will provide some **survey on the use of various function spaces which allow to discuss the robustness of a number of procedures in analysis** (such as reconstruction of smooth functions from their sampling values or atomic decompositions). As we want to point out, the family if $\boldsymbol{L}^p$-spaces is too small to provide good estimates, so some alternative function spaces (such as Wiener amalgam spaces or modulation spaces) will be needed. We will discuss their relevance, mostly in the context of *sampling theory, time-frequency or Gabor analysis and for the theory of Banach frames*.
As time permits we will also describe the situation from the point of view of *coorbit theory*. We will restrict our attention to the setting of function spaces on $\mathbb{R}^d$.

What are Function Spaces good for?

This talk is a contribution to the question in the title. As Yves Meyer (Abel Prize Winner of 2017):
Function Spaces should serve the **description of operators**.
There is a long list of examples, e.g. $\boldsymbol{L}^p$-spaces describing to some extent the behaviour of the Fourier transform (Hausdorff-Young), or the classical **Besov-Triebel-Lizorkin spaces** well suited for Calderon-Zygmund operators or pseudo-differential operators (and wavelets provide the key to understand things better!).
But this application does not have to come first. The **modulation spaces** (introduced in 1983) being a good example. Once their basic properties where established it was realized that they are well suited to describe certain pseudo-differential operators,
e.g. those (underspread channels) which can be used to model *slowly varying channels* in mobile communication.

Which Function Spaces fit to which problems?

So instead of discarding the idea that in the long run function spaces should show their usefulness (in whatever sense) let us try to be more specific. We could ask the following questions:

1. For which families of operators one can say that a particular kind of function spaces is "most appropriate"?
2. Can one always describe the members of these families by atomic decompositions or by the behaviour of some continuous transform (e.g. wavelet transform or STFT)?
3. What kind of mathematical arguments do we have to show that particular function spaces fit best in order to guarantee certain properties of its members?

In this talk we want to concentrate on *robustness issues* (towards imprecise information, e.g. jitter error, model error).

Optimal Function Spaces I

Let us also mention a few general ideas which influence the use of certain function spaces within certain fields. We just list the corresponding properties of the “interesting ones” (or the “most used” ones):

1. First of all those spaces are considered as most suitable for a *particular problem* which give the **strongest results** (in the logical sense), i.e. which allow to describe the largest domain or the smallest target space under given circumstances;
2. Often these *logically optimal* spaces are **complicated objects**, and thus they might be of limited use for applications; learning the technical details for a single application may not be worthwhile;

Optimal Function Spaces II

1. Even if the spaces are easily described in mathematical term it may be quite *difficult to check* in concrete cases whether a given function or distribution satisfies (e.g. an infinity of conditions or asymptotic behaviour);
2. So sometimes *simple spaces* allowing good suboptimal statements might be used much more often;
3. Some function spaces are not just very useful for one application but for a **number of different** (maybe related) **settings**. Then it makes even more sense to learn the detailed properties of such a space.

Optimal Function Spaces, Why are they good?

Given certain *useful spaces* we can reverse the view-point:

1. What are the applications where these spaces fit well?
2. Which of their properties makes them so useful?
3. Is it earning relevance from the relevance of concrete applications?
4. At a more technical level: Which feature and which key observations provide these “good spaces” with their convenient properties?

Of course such questions should be kept in the back of your mind and we will only be able to offer some indications for concrete cases.

Suboptimal Spaces (private opinion)

Probably everybody here knows some cases of spaces which are just to complicated, or not well suited to describe a given operator (often the motivatin to introduce new and better ones):

- The **Schwartz-Bruhat space** $\mathcal{S}(G)$ over LCA groups is certainly a (nuclear Frechet) space having almost all the nice properties of $\mathcal{S}(\mathbb{R}^d)$ but is a terrible complicated object;
- **Quasimeasures** have been introduced specifically in order to describe translation invariant linear operators;
- **Transformable measures** are a specific class of Radon measures which still have a Fourier transform which is a measure;

Often these spaces are *difficult to describe*, they are only applicable to a *small range of problems* or simply did not find widespread distribution for other reasons.

Notations and Conventions

Let us collect here the normalizations of the Fourier transform and relevant transformations of function spaces.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t}\, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega}\, d\omega, \tag{2}$$

which is valid at least for those continuous, integrable functions which have a Fourier transform $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

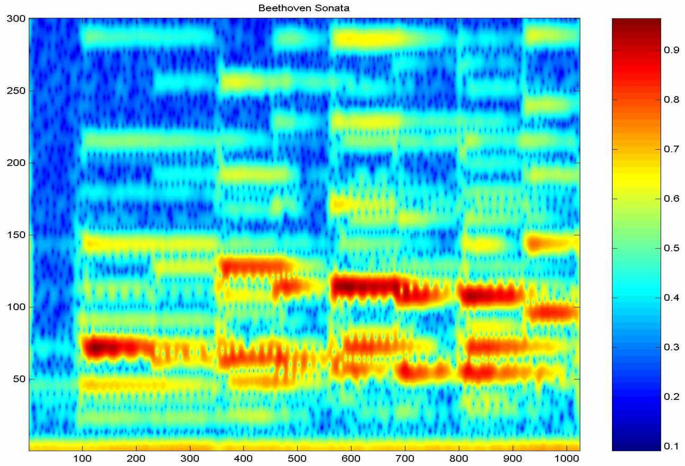
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

Typical Musical STFT: Beethoven Piano Sonata



The Classical Shannon Theorem

For bandlimited functions $f \in \boldsymbol{L}^2(\mathbb{R})$, with $\mathrm{spec}(f) := \mathrm{supp}(\widehat{f}) \subset [-1/2, 1/2]$ we have the well-known reconstruction using the SINC-function $\mathrm{sinc} := \mathcal{F}^{-1}(box)$

$$f = \sum_{k \in \mathbb{Z}} f(k) T_k \,\mathrm{sinc}\,. \tag{3}$$

Good and bad properties of this expansion:

1. Since $f = f * \mathrm{sinc}$ in this situation we have $f(k) = \langle f, T_k \,\mathrm{sinc} \rangle$, and thus (3) is just the expansion of f with respect to an ONB for the underlying Hilbert space;
2. The formula is valid for band-limited $\boldsymbol{L}^p$-functions, for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$.
3. For $p \to 1$ (or $\to \infty$) the constants for the synthesis operator $(c_k) \to \sum_k c_k T_k \,\mathrm{sinc}$ from ℓ^p to $\boldsymbol{L}^p(\mathbb{R}^d)$ explode.

The Practical Version of Shannon's Theorem

The poor decay of the sinc-function, resulting in the fact that sinc $\notin \boldsymbol{L}^1(\mathbb{R})$ also spoils the *locality* of the representation (3) and therefore also engineering books suggest to allow a bit of oversampling, i.e. sample at some rate $\alpha < 1$ which allows to obtain a reconstruction of a very similar nature, namely

$$f = \sum_{k \in \mathbb{Z}} f(\alpha k) T_{\alpha k} g, \tag{4}$$

for some $\alpha < 1$, where $g \in \boldsymbol{L}^1(\mathbb{R})$ can be any function with $\hat{g}(\omega) = 1$ on $[-1/2, 1/2]$ but with $\mathrm{supp}(\hat{g}) \subset \alpha^{-1}[-1/2, 1/2]$. Alternatively one could use the reconstruction formula

$$f = \sum_{k \in \mathbb{Z}} f(k) T_k g, \qquad (5) \tag{5}$$

but only for functions with $\mathrm{spec}(f) \subset \alpha \cdot [-1/2, 1/2]$.

The Practical Version of Shannon's Theorem II

If we compare the two versions, we observe that:

- The first version (3) appears to be perfect, but only from the (*very narrow*) perspective of Hilbert spaces: ONB!
- The second version (4) requires a higher sampling density, hence the shifted building blocks are not anymore an ONB, but might be redundant (just think of the case $\alpha = 0.5$!);
- The third version (5) restricts the representation (atomic decomposition) to certain subspaces of band-limited functions.

There are also some clear benefits:

- by requiring enough smoothness of $\hat{g}$ one can use the theorem for families of weighted $\boldsymbol{L}^p$-spaces;
- there is good locality of the representation!
- there is automatically some robustness towards e.g. jitter error, for the family of spaces involved.

Perturbation results: Kadets Theorem

Early in the history of sampling theory the question came up: Can we still reconstruct, if the samples are taken not exactly at a regular grid (with sufficiently high density), but instead at (known) points near those given lattice points.
The classical answer to this problem is the Kadets $1/4$ Theorem. In a functional analytic description it tells us that the perturbation of the integer lattice points by at most $1/4$ guarantees that the resulting system of shifted SINC functions is at least a *Riesz basis* for the space of band-limited functions. Consequently one can find a biorthogonal system (bounded in the $\boldsymbol{L}^2(\mathbb{R})$-sense) which allows to reconstruct f from the (jittered) sampling values.
But what happens if $f \notin \boldsymbol{L}^2(\mathbb{R})$ or if it is not strictly band-limited. In those cases one has to be careful. But the modified versions of Shannon's Theorem are certainly more robust.

The story of irregular sampling

The story of *irregular sampling theory* allows to treat much more general situations. Given the sampling values of band-limited functions (on $\mathbb{R}^d$ or LCA groups), together with known a-priori information about the *spectral support* of f allows to reconstruct f using iterative methods.
The typical theorems describes the situation as follows: Assume that the *maximal gap size* is small enough (compared to the size of the spectrum, comparable to a Nyquist criterion) then various iterative algorithms can be described which reconstruct band-limited functions f in any member of a family of Banach spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in a linear way (the limit of a sequence of interations). The convergence (at a geometric reat) can be described by means of the function space norm on $\boldsymbol{B}$, and the guaranteed speed of convergence improves with the density of sampling information.

Connections to Frame Theory

The redundancy issue above is of course closely related to the theory of *frames*. Mostly because there is meanwhile a huge body of literature let us mention a few related concepts:

- Classical: frames for Hilbert spaces (or subspaces);
- atomic systems, quasi-frames (e.g. Shidong Li, Ogawa,.);
- Banach frames (K. Gröchenig);
- g-frames (W. Sun);
- continuous frames (long history, preceding the discrete case!).

Without discussing it here a hidden agenda of this talk will be the emphasis on *unconditional Banach frames for families of Banach spaces (of distributions)*.

Irregular sampling in spline-type spaces

The *irregular sampling problem* for *shift-invariant spaces* (also called *spline-type spaces*) is a natural modification of the band-limited case.
One may think of the reconstruction of functions in spaces of cubic polynomials as a non-trival propotypical example.

But let us first describe the situation, noting that in the context of wavelet MRA construction they arise as level zero spaces $\boldsymbol{V}_0$, with a Riesz basis of integer shifts of some "father wavelet", or *generator* φ of the *principle shift invariant space* $\boldsymbol{V}_{\varphi,\Lambda}$, which is the set of translates of φ along some lattice in $\mathbb{R}^d$:

$$V_{\varphi,\Lambda} = \{f = \sum_{\lambda\in\Lambda} a_\lambda T_\lambda \varphi, (a_\lambda) \in \ell^2(\Lambda)\} \qquad (6)$$

Irregular Sampling in spline-type spaces II

Again we have similar (linear and iterative) reconstruction techniques once the density is suitable, given the atom φ or at least some good *qualitative* description of its decay and smoothness. What is (technically) important is the fact, that the set of translates is no only describing the *Hilbert space* situation, i.e. cubic splines in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but that this situation **extends** to a family of function spaces, e.g. weighted $\boldsymbol{L}^p$-spaces on $\mathbb{R}^d$ with polynomial weights.

For example we expect (discussing only the unweighted case, but for the full range $p \in [1, \infty)$) that the closure of the finite linear combinations of translates of φ in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ coincides with those functions in $\boldsymbol{L}^p(\mathbb{R}^d)$ which can be represented as unconditionally convergent infinite sums with ℓ^p-coefficients.

Irregular Sampling in spline-type spaces III

The typical questions arising in this context are then:

- Which qualities of φ allow to formulate irregular sampling theorems (e.g. iterative, linear reconstruction theorems) which are valid for a certain range/family of function spaces, i.e. how can one derive *sufficient conditions for recovery* based on smoothness and decay of φ?
- Can one estimate e.g. the **jitter error** (uniformly over the family of function spaces), given a certain maximal jitter error at each of the points independently.
- Assume there is a **model error**, i.e. the building block φ is not known exactly, but only approximately. How does this influence the result of these reconstruction methods?

The relevant space: Wiener's Algebra

It has turned out (there is meanwhile a long list of publications on the subject) that the most natural and simple condition on φ which allows to provide such estimates is in terms of Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$.
This space (of bounded and continuous) functions on $\mathbb{R}^d$ can be described roughly as the linear space of all *absolutely Rieman integrable functions*, resp. the space of all continuous functions with finite upper Riemannian sum.

A sufficient condition for a continuous function f on $\mathbb{R}^d$ is:

$$|f(x)| \leq C(1+|x|)^{-(d+\varepsilon)}, \quad x \in \mathbb{R}^d.$$

The relevant space: Wiener's Algebra II

Among the main reasons, why Wiener's algebra is so important, we can identify these two most important ones:

1. The **atomic decomposition**: Every $f \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)$ is the absolutely convergent sum of functions (in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$) of functions with support in sets of the form of $x_n + Q$ (e.g. in the unit cube $Q = [0,1]^d$);
2. The **convolution relations** between the more general *Wiener amalgam spaces* and Wiener's algebra, e.g.

$$\boldsymbol{W}(\boldsymbol{M}, \ell^p) * \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^p).$$

Recalling the concept of Wiener Amalgam Spaces

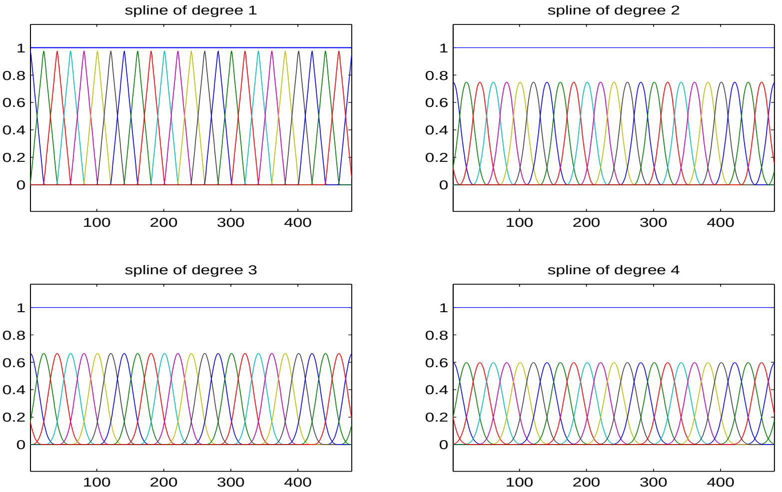
Wiener amalgam spaces are a generally useful family of spaces with a wide range of applications in analysis. The main motivation for the introduction of these spaces came from the observations that the non-inclusion results between spaes $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for different values of p are either of *local* or of *global* nature. Hence it makes sense to separate these to properties using BUPUs.

Definition

A bounded family $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathbb{R}^d$ is called a regular **Uniform Partition of Unity** if $\psi_n = T_{\alpha n}\psi_0, n \in \mathbb{Z}^d$, $0 \leq \psi_0 \leq 1$, for some ψ_0 with compact support, and

$$\sum_{n\in\mathbb{Z}^d} \psi_n(x) = \sum_{n\in\mathbb{Z}^d} \psi(x-\alpha n) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Illustration of the B-splines providing BUPUs



Recalling the concept of Wiener Amalgam Spaces II

Note that one can **define the Wiener amalgam space** $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$ by the condition that the sequence $\|f\psi_n\|_{\boldsymbol{B}}$ belongs to $\ell^q(\mathbb{Z}^d)$ and its norm is one of the (many equivalent) norms on this space.

Different BUPUs define the same space and equivalent norms. Moreover, for $1 \leq q \leq \infty$ one has Banach spaces, with natural inclusion, duality and interpolation properties.
Many known function spaces are also Wiener amalgam spaces:

- $\boldsymbol{L}^p(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{L}^p, \ell^p)$, same for weighted spaces;
- $\mathcal{H}_s(\mathbb{R}^d)$ (the Sobolev space) satisfies the so-called ℓ^2-puzzle condition (P. Tchamitchian): $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{H}_s, \ell^2)$, and consequently for $s > d/2$ (Sobolev embedding) the pointwise multipliers (V. Mazya) equal $\boldsymbol{W}(\mathcal{H}_s, \ell^\infty)$.

Minimality of Wiener's algebra

The Wiener amalgam spaces are essentially a generalization of the original family $\boldsymbol{W}(\boldsymbol{L}^p, \boldsymbol{\ell}^q)$, with local component $\boldsymbol{L}^p$ and global q-summability of the sequence of local $\boldsymbol{L}^p$ norms.
In contrast to the "scale" of spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right), 1 \leq p \leq \infty$ which do *not allow for any non-trivial inclusion relations* we have nice (and strict) inclusion relations for $p_1 \geq p_2$ and $q_1 \leq q_2$:

$$\boldsymbol{W}(\boldsymbol{L}^{p_1}, \boldsymbol{\ell}^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \boldsymbol{\ell}^{q_2}).$$

Hence $\boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ is the smallest among them, and $\boldsymbol{W}(\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)$ is the largest among them. The closure of the space of test functions, or also of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ is just *Wiener's algebra* $\left(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$, which was one of Hans Reiter's list *Segal algebras*. It can also be characterized as the smallest of all *solid* Segal algebras.

Introducing Modulation Spaces

Having the possibility to define Wiener amalgam spaces with $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ (the Fourier image of $\boldsymbol{L}^p(\mathbb{R}^d)$ in the sense of distributions) as a local component allowed to introduce **modulation spaces** in analogy to *Besov spaces*, replacing more or less the dyadic decompositions on the Fourier transform side by uniform ones.

Formally one can define the (unweighted) modulation spaces as

$$\boldsymbol{M}^{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)\right). \tag{7}$$

or more generally the now classical modulation spaces

$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q_{v_s})\right). \tag{8}$$

Fourier invariant modulation spaces

It is an interesting variant of the classical Hausdorff-Young theorem to observe that one has

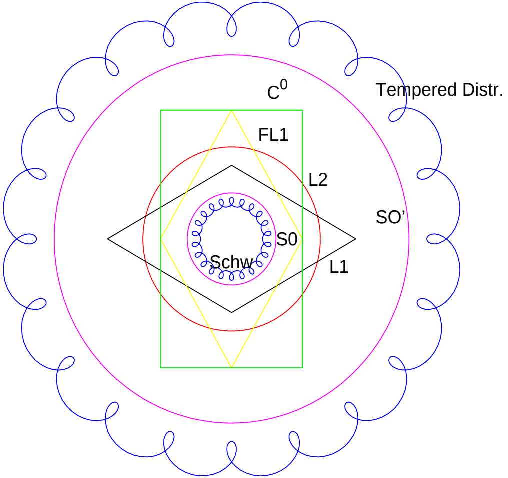
Theorem

- *For* $1 \leq r \leq p \leq \infty$ *one has*
$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{F}^p, \ell^r)) \subseteq \boldsymbol{W}(\boldsymbol{F}^r, \ell^p);$$
- *and as a consequence for* $1 \leq p, q \leq 2$*:*
$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)) \subseteq \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'}).$$

The Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$

Within the family of Banach spaces of (tempered) distributions of the form $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$ we have natural inclusions. The smallest in this family is the space $M_0^{1,1}(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$, which is a *Segal algebra* and the smallest non-trivial Banach space isometrically invariant under time-frequency shifts.
It is Fourier invariant, as well as all the spaces $\boldsymbol{M}^p := \boldsymbol{M}^{p,p}$, with $1 \leq q \leq \infty$. This last mentioned space $\boldsymbol{M}^\infty(\mathbb{R}^d)$ coincides with $\boldsymbol{S}_0'(\mathbb{R}^d)$, the dual of $\boldsymbol{S}_0(\mathbb{R}^d)$, and is the largest TF-invariant Banach space.
In the middle we have the space $\boldsymbol{M}^2 := \boldsymbol{M}^{2,2} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2)$.
Together the triple of space $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ forms a so-called **Banach Gelfand Triple** which is highly useful for many applications (especially TF-analysis).

My favorite Function Space plot



Usefulness of $\boldsymbol{S}_0(\mathbb{R}^d)$ in Fourier Analysis

Most consequences result form the following inclusion relations:

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d);$$

$$\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d);$$

$$(\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d);$$

$$(\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d);$$

1. $\boldsymbol{S}_0(\mathbb{R}^d)$ is a valid domain of Poisson's formula;
2. all the classical Fourier summability kernels are in $\boldsymbol{S}_0(\mathbb{R}^d)$;
3. modelling of stationary stochastic processes;
4. the elements $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ are the natural building blocks for Gabor expansions;

Summability kernels from the space $S_0(\mathbb{R}^d)$

First of all we have $\boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$. Its members and their Fourier transforms are integrable (in the sense of Riemann). Consequently Fourier inversion is valid in the pointwise sense:

$$f(t) = \int_{\mathbb{R}^d} \widehat{f}(x) e^{-2\pi i t \cdot s} ds, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

On the other hand we have for a general $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ $\widehat{f} \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$, but not necessarily in $\boldsymbol{L}^1(\mathbb{R}^d)$ so that the inverse Fourier transform may not be feasible as a (Lebesgue) integral. But we can argue that $\widehat{f} \cdot h \in \boldsymbol{L}^1(\mathbb{R}^d)$ (for any $h \in \boldsymbol{S}_0(\mathbb{R}^d)$) and thus we can apply the inverse transform to this pointwise product. Choosing $h(s) = \hat{g}(\rho s), \rho > 0$ for some $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\hat{g}(0) = \int_{\mathbb{R}^d} g(x) dx = 1$ we see that the Fourier inversion recovers $\mathrm{St}_\rho g * f$ (the compressed version of g tending to δ_0), which tends to f in the $\boldsymbol{L}^1$-sense.

Recalling Gabor Expansions

A **Gabor family** $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is an indexed family obtained by applying time-frequency shifts of the form $\pi(\lambda) = M_s T_t$, for $\lambda = (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to a *Gabor atom* g (in $\boldsymbol{S}_0(\mathbb{R}^d)/\boldsymbol{L}^2(\mathbb{R}^d)$).
We are mostly interested in *Gabor frames* or (for mobile communication) in *Gabor Riesz bases*.
A function or distribution f has a **Gabor expansion** if

$$f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g = \sum_{\lambda\in\Lambda} c_\lambda g_\lambda \qquad (9)$$

for a suitable family of complex-valued coefficients $(c_\lambda)_{\lambda\in\Lambda}$.
A key player in this context is the Gabor frame-like operator:

$$S := S_{g,\Lambda} : f \mapsto \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda .$$

(g,Λ) generates a **Gabor frame** iff S is invertible on $\boldsymbol{L}^2$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis I

First we observe that (as a consequence of the abstract coorbit theory, developed together with K. Gröchenig around 1988/89):

Theorem

Given $g \in \mathbf{S}_0(\mathbb{R}^d)$ there exists $\varepsilon > 0$ such that for every ε-dense family $(\lambda_i)_{i\in I}$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, i.e. with $\bigcup_{i\in I} B_\varepsilon(\lambda_i) = \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ (with appropriate adaptive weights) forms an (irregular) Gabor frame for $\mathbf{L}^2(\mathbb{R}^d)$.
There is a bounded linear mapping from $\mathbf{L}^2(\mathbb{R}^d)$ into (a weighted version of) $\ell^2(I)$ which provides the appropriate coefficients for a Gabor expansion of any $\mathbf{L}^2(\mathbb{R}^d)$.
The restriction of this linear mapping to $\mathbf{S}_0(\mathbb{R}^d) \subset \mathbf{L}^2(\mathbb{R}^d)$ provides coefficients in the corresponding ℓ^1-space. On the other hand this mapping extends in a unique (w^*-w^*--continous) sense to $\mathbf{S}_0'(\mathbb{R}^d)$, providing coefficients in ℓ^∞.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis II

Although the invertibility of $S = S_{g,\Lambda}$ is considered a priori only as operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ it is of great interest to know whether it is also invertible on other spaces, e.g. on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.
The fact that one has for the case of a lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^{2d}$:

$$\pi(\lambda) \circ S_{g,\Lambda} = S_{g,\Lambda} \circ \pi(\lambda), \quad \lambda \in \Lambda,$$

implies that the canonical dual frame given as the family

$$(S^{-1} g_\lambda) = \pi(\lambda)(S^{-1} g), \quad \lambda \in \Lambda,$$

is again a *regular Gabor frame*, using the *dual Gabor atom* $\tilde{g} := S^{-1} g$ as Gabor atom.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis III

In fact, it is not difficult to check that for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the frame-operator is then also a bounded operator on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, still the invertibility of the restriction of the invertible frame operator $S_{g,\Lambda}$ to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ might not be invertible on that dense subspace. However, using deep Banach algebra methods the following result (a non-commutative version of Wiener's Lemma) has been shown by Gröchenig and Leinert.

Theorem

If (g, Λ) generates a Gabor frame for some $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ then the frame operator is also invertible on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, or equivalently, also the dual Gabor atom $\tilde{g} = S_{g,\Lambda}^{-1}(g) \in \boldsymbol{S}_0(\mathbb{R}^d)$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis IV

In both cases an important question is the so-called Bessel property:

Lemma

Given a compact set of non-singular matrices $\mathbf{M}$ *and* $g \in \mathbf{S}_0(\mathbb{R}^d)$ *there exists some constant* $C > 0$ *such that*

$$\sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leq C \|f\|_2, \quad \forall f \in \mathbf{L}^2(\mathbb{R}^d)$$

for any lattice of the form $\mathbf{A} * \mathbb{Z}^{2d}$, *with* $\mathbf{A} \in \mathbf{M}$.

If normalized suitable one can take the limit $\Lambda \to \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, i.e. lattices which are more and more refined and then the normalized dual windows will converge in $\mathbf{S}_0(\mathbb{R}^d)$ to g.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis V

This uniform is a minor building block for the question of "varying the lattice": What can we say about the dual Gabor atoms $\tilde{g}$ for slightly different lattices? (e.g. a rational approximation of a general lattice). Here only for $\boldsymbol{S}_0(\mathbb{R}^d)$ we can claim that the dual window depends (in a reasonable way) continuous on the lattice (and for simple reasons on the window, if continuity is expressed in the $\boldsymbol{S}_0(\mathbb{R}^d)$-norm). We have (HGFei + N. Kaiblinger):

Theorem

The set $\{(\boldsymbol{A}, g) \,|\, (g, \Lambda)$ forms a Gabor frame$\}$ is an open subset of $\boldsymbol{S}_0(\mathbb{R}^d) \times GL(n, \mathbb{R})\}$ and the mapping $(\boldsymbol{A}, g) \to \tilde{g} = S_{g,\Lambda}^{-1}(g)$ is a continuous mapping into $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis VI

It is important to have continuous dependence of dual Gabor window in the norm of $\boldsymbol{S}_0(\mathbb{R}^d)$ and not just in the $\boldsymbol{L}^2$-norm, because it may happen, that one has a computationally efficient way to compute the dual window in a specific case (e.g. rational lattices), and then one would like to use that *computable* dual window $\tilde{g}_a$ as a replacement for the true dual window $\tilde{g}$. Of course, one would like to be sure that the *approximate recovery operator*

$$S_a : f \mapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \pi(\lambda) \tilde{g}_a$$

is close to the identity operator, in the operator norm sense (on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, for example), and for this it is not enough to have a good $\boldsymbol{L}^2$-approximation of $\tilde{g}$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis VII

In practice it is usually not possible to invert an operator exactly using numerical methods. Since Gabor analysis can be done over general LCA groups we can formulate properties of Gabor families also over finite Abelian groups such as $\mathbb{Z}_N$, for $N \in \mathbb{N}$.
A result by N. Kaiblinger gives the following result:

Theorem

For any pair $(g, a\mathbb{Z} \times b\mathbb{Z})$, with $g \in \boldsymbol{S}_0(\mathbb{R})$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and a finite Gabor family over $\mathbb{Z}_N$ (i.e. a sampled version of g in $\mathbb{C}^N$, and divisors a_N, b_N of N) such that the piecewise linear interpolation $\tilde{g}_a$ of the computed discrete dual Gabor atom in $\mathbb{C}^N$ satisfies

$$\|\tilde{g} - \tilde{g}_a\|_{\boldsymbol{S}_0} \leq \varepsilon.$$

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis VIII

The control of *irregular Gabor families* for general discrete (well-spread) sets $\Lambda \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with respect to modification of the sampling set comes in two versions (the first as a consequence of coorbit theory, the second proved by Ascensi/F/Kaiblinger):

- Assume that (g, Λ) generates a Gabor frame, with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then for $\varepsilon > 0$ there exists $\delta > 0$ such that for any family Λ' with $|\lambda - \lambda'| < \delta, \forall \lambda \in \Lambda$ implies

$$|||S_{g,\Lambda} - S_{g,\Lambda'}|||_{\boldsymbol{L}^2(\mathbb{R}^d)} < \varepsilon.$$

- A similar statement is valid for $\Lambda' = \boldsymbol{B} * \Lambda$ with

$$\|\boldsymbol{B} - Id_{2N}\| < \delta.$$

Illustrations 1

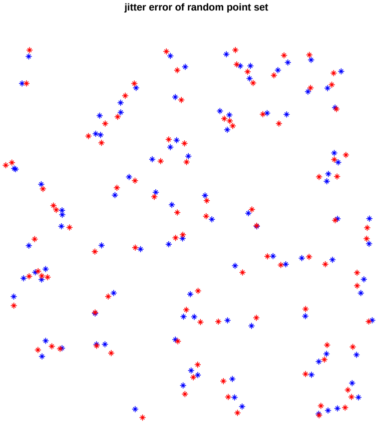


Figure: red points: jitter error

Illustrations 2

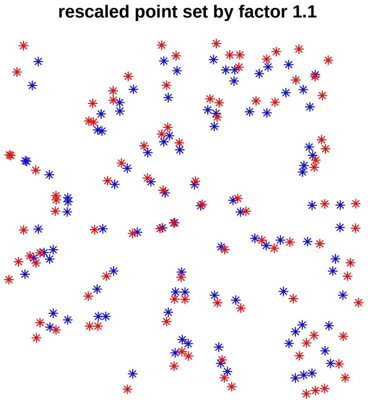


Figure: red points: dilation of point set by 1.1

Illustrations 3

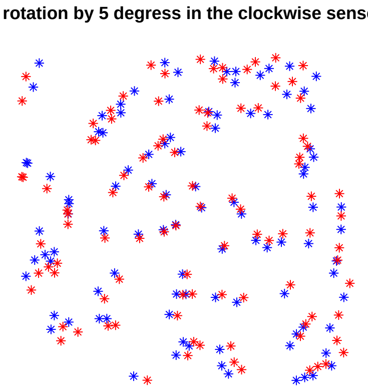


Figure: red points: rotation of the point set by 5 degress

Further Characterizations of $\boldsymbol{S}_0(\mathbb{R}^d)$

- atomic decompositions, e.g. using Gaussians;
- modern approach: integrability of the STFT;
- "algebraic irreducibility": any element can be use to rebuild $\boldsymbol{S}_0(\mathbb{R}^d)$ via absolutely convergent atomic series.

Characterization of $S_0(\mathbb{R}^d)$ via STFT

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Characterization of $S_0'(\mathbb{R}^d)$ via STFT

The short-time Fourier transform with respect to any Schwartz window $g \in \mathcal{S}(\mathbb{R}^d)$ can be defined for any tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$. It is not difficult to show that they are all growing at most like some polynomial, e.g. $(1+|\lambda|^2)^k$ for some $k \in \mathbb{N}$. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ we have a natural embedding from $S_0'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

Lemma

A tempered distribution defines a bounded, linear functional on $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$, i.e. defines an element of $S_0'(\mathbb{R}^d)$ if and only if its STFT is a bounded (continuous) function.

Norm convergence in $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is equivalent to uniform convergence of the corresponding STFTs. On the other hand w^*-convergence of a net (σ_α) to some limit $\sigma_0 \in S_0'(\mathbb{R}^d)$ is the same as uniform convergence over compact subsets.

Atomic Characterization of $S_0(\mathbb{R}^d)$

One important characterization of $S_0(\mathbb{R}^d)$ is via (absolutely convergent) **atomic compositions** of its elements.

Definition

$$\mathcal{S}_{atom} := \{f \in \boldsymbol{L}^2(\mathbb{R}^d) \mid f = \sum_{k=1}^{\infty} c_k \pi(\lambda_k) g_0 \,,\, \sum_{k=1}^{\infty} |c_k| < \infty\}$$

It is a Banach space with the quotient norm

$$\|f\|_{atom} := \inf\{\sum_k |c_k|, \text{over all admiss. representations of } f\}.$$

Using this criterion, with $g = g_0$ (the Fourier invariant Gauss function) it is quite obvious that this Banach space is isometrically invariant under the Fourier transform.

Further relevant application areas

- Kernel theorems: e.g. Characterization of linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$ are characterized by their corresponding distributional "kernel" $K(x,y) \in \boldsymbol{S}_0'(\mathbb{R}^d)$;
- The *spreading representation* of operators, pseudo-diff. ops.;
- Characterization of (Fourier) multipliers, e.g. from $\boldsymbol{L}^p(\mathbb{R}^d)$ into $\boldsymbol{L}^q(\mathbb{R}^d)$ within $\boldsymbol{S}_0'(\mathbb{R}^d)$;
- BANACH GELFAND TRIPLES
- FOURIER STANDARD SPACES (TF-invariant spaces between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$).

For details on these subjects see the various talks by the author at `www.nuhag.eu/talks` and a forthcoming book with Georg Zimmermann.

What is the key property of $S_0(\mathbb{R}^d)$?

Among the many properties of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ at appears that the **minimality property** is the most important one. One formulation in the context of Schwartz theory is the following one:

Lemma

$\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the **smallest** *among all Banach spaces which contain $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and are isometric invariant with respect to TF-shifts $\pi(\lambda) = M_s T_t$, for $\lambda = (t, s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.*

Alternatively it can be described as the **smallest** *among all* **Segal algebras** *(dense Banach ideals in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$) which are isometrically invariant under multiplications with pure frequencies.*

This is the reason for the choice of the symbol $\boldsymbol{S}_0$!

Similar results for the weighted cases

It is a natural next step to ask, if there is also a smallest subspace of $\mathbf{L}^2(\mathbb{R}^d)$ among all the Banach spaces $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ containing e.g. $\mathcal{S}(\mathbb{R}^d)$ and with the property that there exist *submultiplicative* weight functions w_1, w_2 such that for all $t, s \in \mathbb{R}^d$ one has:

$$\|T_t f\| \leq w_1(t)\|f\|_{\mathbf{B}}, \quad \forall f \in \mathbf{B},$$

$$\|M_s f\| \leq w_2(s)\|f\|_{\mathbf{B}}, \quad \forall f \in \mathbf{B},$$

Lemma

The smallest among all such Banach spaces is the Wiener amalgam space $\mathbf{W}(\mathcal{F}\mathbf{L}^1_{w_2}, \ell^1_{w_1})$.

Outlook on coorbit theory

There are very similar structures in the context of the so-called **coorbit theory** (developed jointly with K. Gröchenig).

Instead of time-frequency shift which represent an *irreducible projective representation* of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ one starts there from a **unitary, irreducible and integrable representation** π of some locally compact (non-Ablian!) group G on an abstract Hilbert space $\mathcal{H}$ and builds up a very similar theory. Again Wiener amalgam spaces on G play a crucial role for the analysis.

The STFT is replaced by the *voice transform*

$$V_g(f)(x) := \langle f, \pi(\lambda)g \rangle, \quad f, g \in \mathcal{H}.$$

Outlook on coorbit theory II

The minimal spaces in this context are then of the form

$$\boldsymbol{Co}(\boldsymbol{L}^1_w) := \{f \in \mathcal{H} \mid V_g(f) \in \boldsymbol{L}^1(G)\}$$

with the norm $\|f\| := \|V_g(f)\|_{\boldsymbol{L}^1_w(G)}$ (for a weight on G).
For the case of the $ax+b$-group (of affine transformations of the real line) this transform comes out to coincide with the **wavelet transform** (with window g).
The resulting coorbit spaces with respect to mixed-norm weighted function spaces on that group turns out to contain the *classical* function spaces, namely the **Besov-Triebel-Lizorkin spaces**.
The minimal ones are then homogeneous Besov space with parameters $p = 1 = q$ and well chosen $s > 0$.

Publications related to this context

[4] K. Gröchenigs book is a standard book on time-frequency analysis. The *Segal algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is described there as $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$, the modulation space corresponding to the parameters $p = 1 = q$.

Feichtinger, Hans G.: Choosing Function Spaces in Harmonic Analysis [2]: Features some ideas concerning construction principles of function spaces.

Cordero, Elena;Feichtinger, Hans G.;Luef, Franz: Banach Gelfand triples for Gabor analysis [1] describes a simple approach to the theory of Banach Gelfand Triples, with $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ as an imporant special case.

Feichtinger, Hans G.: Thoughts on Numerical and Conceptual Harmonic Analysis [3] describes ideas about the connection between finite discrete groups approximating the continuous case.

References

Elena Cordero, Hans G. Feichtinger, and Franz Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

Hans G. Feichtinger.
Choosing Function Spaces in Harmonic Analysis, volume 4 of *The February Fourier Talks at the Norbert Wiener Center, Appl. Numer. Harmon. Anal.*, pages 65–101.
Birkhäuser/Springer, Cham, 2015.

Hans G. Feichtinger.
Thoughts on Numerical and Conceptual Harmonic Analysis.
In Akram Aldroubi, Carlos Cabrelli, Stéphane Jaffard, and Ursula Molter, editors, *New Trends in Applied Harmonic Analysis. Sparse Representations, Compressed Sensing, and Multifractal Analysis*, Applied and Numerical Harmonic Analysis., pages 1–28. Birkhäuser, 2016.

Karlheinz Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Thank you for your attention!

If you are interested in the subject maybe our time-frequency conference in Strobl near Salzburg, June 3-9th, 2018 is of interst to you: see

www.nuhag.eu/strobl18

The idea of function space design

The above considerations suggest to think about the possibility for the **design of families of function spaces**, the motivation being **not just** this one:

1. doing things because they can be done;
2. defining multi-parameter spaces as extension of the "classical ones" and demonstrating analogous properties;
3. hope that somebody will finally find applications.

but rather describe the setting (e.g. operators, differential equations) where currently available function spaces are not providing satisfactory results and where new spaces with *certain specified properties* would be useful. Then others might do such *"constructions on demand"* or demonstrate that such objects do not exist!

Comparison with user-friendly software

If I look at software systems, e.g. modern electronic banking systems (I changed my bank last year for exactly this reason) or portals/machines selling e.g. concert or train tickets (OEBB has installed new ones) it is clear that the "good ones" (compared with the classical/old ones) have many advantages.

1. they offer more possibilities;
2. they are easier to use (more intuitive), less learning required;
3. they are more flexible with respect to (natural, but formally incorrect) use (hopefully), e.g. order of input;

I suggest, that **as a community we should start thinking of producing consumer reports** concerning function spaces. Which ones are good for which purpose? In developing criteria (and discussing them) we make progress towards a better use of the different spaces and awareness what to look for!