New Perspectives in the Theory of Function Spaces and their Applications
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Hardy spaces associated with semi-groups of linear operators

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Hardy space $H^p(\mathbb{R}^N)$

$H^p(\mathbb{R}^N)$ comes from study holomorphic functions in $\mathbb{R}^{2+}$

For a function $u(x + iy)$ defined in upper half-plane $x \in \mathbb{R}, y > 0$, we define the non-tangential maximal function

$$u^*(x) = \sup_{|x - x'| < y} |u(x' + iy)|$$

$\{(x', y) : |x - x'| < y\}$
A real-valued harmonic function \( u(x, y) = u(x + iy) \), \( y > 0 \), is a real part of a holomorphic function \( F(z) = u(z) + iv(z) \) which satisfies (for fixed \( 0 < p < \infty \))

\[
\sup_{y>0} \int_{\mathbb{R}} |F(x + iy)|^p \, dx < \infty \quad (L^p)
\]

if and only if

\[
u^*(x) = \sup_{|x-x'|<y} |u(x' + iy)|
\]

belongs to \( L^p(\mathbb{R}) \).

holomorphy \(\leadsto\) system of conjugate harmonic functions:

\[
u = (u_0, u_1, \ldots, u_N) \in (CR), u_j = u_j(x_0, x_1, \ldots, x_N), x_0 > 0, \text{ if}
\]

\[
d_{x_j}u_i = d_{x_i}u_j, \quad \sum_{j=0}^{N} d_{x_j}u_j = 0, \quad \text{(CR)}
\]

\[
\sup_{x_0 > 0} \int_{\mathbb{R}^N} |u(x_0, x_1, \ldots, x_N)|^p \, dx_1 dx_2 \ldots dx_N < \infty. \quad \text{(L^p)}
\]

1 \(- \frac{1}{N} < p < \infty, u_0(x_0, x_1, \ldots, x_N)\)-harmonic, then \(\exists u = (u_0, u_1, \ldots, u_N)\) satisfying (CR) and (L^p) iff

\[
u_0^*(x) = \sup_{|x-x'| < x_0} |u_0(x_0, x')| \in L^p(\mathbb{R}^N).
\]

holomorphy $\Rightarrow$ system of conjugate harmonic functions:

$u = (u_0, u_1, \ldots, u_N) \in (CR), u_j = u_j(x_0, x_1, \ldots, x_N), x_0 > 0,$ if

$$\partial_{x_j} u_i = \partial_{x_i} u_j, \quad \sum_{j=0}^{N} \partial_{x_j} u_j = 0, \quad (CR)$$

$$\sup_{x_0 > 0} \int_{\mathbb{R}^N} |u(x_0, x_1, \ldots, x_N)|^p \, dx_1 dx_2 \ldots dx_N < \infty. \quad (L^p)$$

$1 - \frac{1}{N} < p < \infty, u_0(x_0, x_1, \ldots, x_N)$-harmonic, then $\exists u = (u_0, u_1, \ldots, u_N)$ satisfying (CR) and (L$^p$) iff

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Holomorphy $\rightsquigarrow$ system of conjugate harmonic functions:

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\[
u_0^*(x) = \sup_{|x-x'|<x_0} |u_0(x_0, x')| \in L^p(\mathbb{R}^N).
\]
Then $\lim_{x_0 \to 0} u_0(x_0, x)$ exists in the sense of distributions and:

$$H^p(\mathbb{R}^N) = \{ f : f = \lim_{x_0 \to 0} u_0(x_0, x), (u_0, u_1, ..., u_N) \in (CR) \cap (L^p) \}$$

Several equivalent definitions of $H^p(\mathbb{R}^N)$

by means of

- the Poisson maximal function
- other maximal functions including the heat maximal function
- Riesz transforms
- square functions
- atoms (Coifman (Studia Math. 1974), Latter (Studia Math. 1978))
Then \( \lim_{x_0 \to 0} u_0(x_0, x) \) exists in the sense of distributions and:

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maximal functions characterizations

\( f \in H^1(\mathbb{R}^n) \) if and only if

\[
\| f \|_{H^1,\text{heat}} = \| \sup_{t > 0} |e^{t\Delta} f(x)| \|_{L^1(\mathbb{R}^n, dx)} < \infty
\]

equivalently,

\[
\| f \|_{H^1,\text{Poisson}} = \| \sup_{t > 0} |e^{-t\sqrt{-\Delta}} f(x)| \|_{L^1(\mathbb{R}^n, dx)} < \infty
\]

other maximal functions
Riesz transform characterization

\[ R_j f(x) = -\partial_j (\Delta)^{-1/2} f(x) = -\frac{i}{c} \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} e^{ix \cdot \xi} \mathcal{F} f(\xi) d\xi \]

\( f \in H^1(\mathbb{R}^n) \) if and only if

\[ \| f \|_{H^1, \text{Riesz}} = \| f \|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \| R_j \|_{L^1(\mathbb{R}^n)} < \infty \]

square function characterization

square function: \( S f(x) = \left( \iint_{|x-y|<t} |t^2 e^{t^2 \Delta} f(y)| dydt \div B(x,t)|t| \right)^{1/2} \)

\( f \in H^1(\mathbb{R}^n) \) if and only if

\[ \| f \|_{H^1, \text{square}} = \| S f \|_{L^1(\mathbb{R}^n)} < \infty \]
atomic decomposition

Fix $1 < q \leq \infty$. A function $a$ is an $(1, q)$-atom for the Hardy space $H^1(\mathbb{R}^d)$ if there is a ball $B$ such that

$\text{supp } a \subset B$ (support condition);

$\|a\|_{L^q} \leq |B|^{\frac{1}{q} - 1}$ (size condition);

$\int a = 0$ (cancellation condition).

Then $\|a\|_{H^1} \leq C_{q,n}$ for every atom $a$.

The atomic norm $\|f\|_{H^1_{\text{atom}}(\mathbb{R}^d)}$ is defined as

$$\|f\|_{H^1_{\text{atom}}(\mathbb{R}^d)} = \inf \sum |\lambda_j|,$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$.

Theorem (Coifman, Latter)

The spaces $H^1(\mathbb{R}^d)$ and $H^1_{\text{atom}}(\mathbb{R}^d)$ coincide and the norms $\|f\|_{H^1(\mathbb{R}^d)}$ and $\|f\|_{H^1_{\text{atom}}(\mathbb{R}^d)}$ are equivalent.
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Remark: generalizations to

- **spaces of homogeneous type**: Coifman and Weiss (Bull. AMS 1977), Macías and Segovia (Advances in Math. 1979), Uchiyama (TAMS 1980);

- **homogeneous nilpotent Lie groups**: Folland and Stein (1982), Christ and Geller (Duke Math. J. 1984). Riesz transform characterization of the space by applying ideas of Uchiyama of constructive Fefferman Stein decomposition of BMO functions (if time permits: application of Christ and Geller result to characterize by relevant Riesz transforms the $H^1$ space associated with the Grushin operator on $\mathbb{R}^{n+1}$)

$$L = -\Delta_x - |x|^2 \frac{\partial^2}{\partial y^2}, \ x \in \mathbb{R}^n, \ y \in \mathbb{R};$$

- **weighted Hardy spaces**: García-Cuerva (Dissertationes Math. 1979), Stromberg and Torchinsky (1989),

- **Hardy spaces for non-doubling measures**: Mauceri, Meda, Sjögren, Vallarino, Fu, Lin, Yang, Yang,...
product Hardy spaces: Chang and R. Fefferman, Merryfiled

Hardy spaces associated with semigroups of linear operators

Take a semigroup $e^{tA}$ of linear operators acting on function spaces on a metric measure space $(X, d, d\mu)$. Consider

$$H^1_{\text{max}} = \{ f : \| f \|_{H^1_{\text{max}}} = \sup_{t>0} \| e^{tA} f \|_{L^1(d\mu)} < \infty \}$$

$$H^1_{\text{square}} = \{ f : \| f \|_{H^1_{\text{max}}} = \sup_{t>0} \| S f \|_{L^1(d\mu)} < \infty \}$$

$$S f(x) = \left( \iint_{d(x,y)<t} |t^2 A e^{t^2 A} f(y)|^2 \frac{dydt}{\mu(B(x,t))|t|} \right)^{1/2}$$

Application of local Hardy spaces

D. Goldberg (Duke Math. J. 1979),
origin: study harmonic functions in \( \{ z = x + iy : 0 < y < 1 \} \)
notion of local (real) Hardy space on \( \mathbb{R}^N \) with \( dx \): fix \( \ell > 0 \),
maximal functions:

\[
\mathcal{M}_\ell f(x) = \sup_{0 < t < \ell} |e^{-t\sqrt{-\Delta}} f(x)|, \quad \|f\|_{h^1_{\ell, \sqrt{-\Delta}}} = \|\mathcal{M}_\ell f\|_{L^1},
\]

\[
M_\ell f(x) = \sup_{0 < t < \ell} |e^{t^2\Delta} f(x)|, \quad \|f\|_{h^1_\ell} = \|M_\ell f\|_{L^1}
\]

\[
M_{(\Delta - \ell^{-2})} f(x) = \sup_{t > 0} |e^{t(\Delta - \ell^{-2})} f(x)|, \quad \|f\|_{H^1_{(\Delta - \ell^{-2})}} = \|M_{(\Delta - \ell^{-2})} f\|_{L^1}
\]

\[
\|\sup_{t > \ell} |e^{t^2(\Delta - \ell^{-2})} f|\|_{L^1} \leq \|f\|_{L^1}
\]

\[
\|\sup_{0 < t < \ell} |e^{t^2(\Delta - \ell^{-2})} f - e^{t^2\Delta} f|\|_{L^1} \leq C \|f\|_{L^1}
\]

local Riesz transforms: \( R_j,\ell f = f \ast R_j,\ell \), \( R_j,\ell = \frac{x_j}{|x|^{N+1}} \chi_{B(0,\ell)}(x) \),

\[
\|f\|_{Riesz, \ell} = \|f\|_{L^1} + \sum \|R_j,\ell f\|_{L^1}
\]
Atomic decomposition: local atoms:

\( a(x) \) is a local \( \ell \)-atom for \( h^1_\ell \) if either

\[
a = |B(y_0, \ell)|^{-1} \chi_{B(y_0, \ell)}
\]

or there is a ball \( B = B(y_0, r), r \leq \ell \), such that

- \( \text{supp } a \subset B \)
- \( \|a\|_{L^\infty} \leq |B|^{-1} \)
- \( \int a = 0 \).

All norms are equivalent and with comparing constants independent of \( \ell \).

Important property of \( h^1_\ell \): If \( \sup |\varphi| \leq 1 \) and \( \sup |\nabla \varphi| \leq \ell^{-1} \), then \( h^1_\ell \ni f \mapsto \varphi f \) is bounded on \( h^1_\ell \) with bound independent of \( \ell \).
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Application to Schrödinger semigroups

\[ T_t = e^{t(\Delta-V)}, \quad V \geq 0 \quad \text{on } \mathbb{R}^d. \]

Joint works with Jacek Zienkiewicz
continued with G.Garrigos, M.Preisner, J.L. Torrea, T.Martinez,...
\[ H_{\Delta - V}^1 = \{ f \in L^1 : \| f \|_{H_{\Delta - V}^1} = \| \sup_{t>0} |e^{t(\Delta - V)} f| \|_{L^1} < \infty \}. \]

The Feynman-Kac formula:

\[ T_t f(x) = \mathbb{E}^x \left( e^{-\int_0^t V(b_s) \, ds} f(b_t) \right), \quad b_t(\omega) \text{ is Brownian motion for } e^{t\Delta}. \]

\[ T_t f(x) = e^{t(\Delta - V)} f(x) = \int_{\mathbb{R}^d} K_t(x, y) f(y) \, dy, \]

\[ 0 \leq K_t(x, y) \leq (4\pi t)^{-d/2} e^{-|x-y|^2/4t}. \]

Does the presence of the potential \( V \) reflect that the factor \( e^{-\int_0^t V(b_s) \, ds} \) gives additional decay for large \( t \) such that the maximal function \( \sup_{t>T} |T_t f(x)| \) is bounded on \( L^1 \) for \( f \) supported in a compact set \( K \)?

If so, can one expect local atoms in our atomic decompositions?

If \( d \geq 3 \), then \( b_t \) is transient. Does the factor give an essential decay for colorredsmall potentials \( V \)?

If \( d = 1, 2 \), then \( b_t \) is recurrent and perhaps even for locally supported potentials the factor gives sufficient decay.
\[ H^1_{\Delta - V} = \{ f \in L^1 : \| f \|_{H^1_{\Delta - V}} = \| \sup_{t>0} |e^{t(\Delta - V)} f| \|_{L^1} < \infty \}. \]

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Does the presence of the potential $V$ reflect that the factor $e^{-\int_0^t V(b_s) \, ds}$ gives additional decay for large $t$ such that the maximal function $\sup_{t>T} |T_t f(x)|$ is bounded on $L^1$ for $f$ supported in a compact set $K$?

If so, can one expect local atoms in our atomic decompositions?

If $d \geq 3$, then $b_t$ is transient. Does the factor give an essential decay for colorredsmall potentials $V$?

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\( H^1_{\Delta-V} = \{ f \in L^1 : \| f \|_{H^1_{\Delta-V}} = \| \sup_{t>0} |e^{t(\Delta-V)} f|\|_{L^1} < \infty \} \).

The Feynman-Kac formula:

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Plan of the talk

- Hardy spaces associated with Schrödinger operators having local atoms in atomic decompositions
- Hardy spaces associated with Schrödinger operators with small potentials (all atoms have cancellation property)
- Hardy spaces for semigroups with upper and lower Gaussian bounds (remark)
- Hardy spaces associated with Grushin operator (Riesz transform characterization)
- Hardy spaces for Dunkl operators (kernels have no Gaussian bounds)
Examples of Hardy spaces for Schrödinger operators with local atoms.

Let $Q = \{Q_j\}_{j=1}^{\infty}$ be a family of closed cubes in $\mathbb{R}^d$ with disjoint interiors such that:

1. $\mathbb{R}^d \setminus \bigcup_{j=1}^{\infty} Q_j$ is of Lebesgue measure zero;
2. If $Q_i, Q_j \in Q$ are neighbors, then diameters $d(Q_i)$ and $d(Q_j)$ are comparable.

Then we may take $\beta > 0$ small such that the family $\{Q^*\}_{j=1}^{\infty}$ has finite covering property, where $Q^* = (1 + \beta)Q$.

A function $a$ is an $H^1_Q$-atom if there exists $Q \in Q$ such that

- either $a = \frac{1}{|Q|} 1_Q$
- or $a$ is the classical atom with support contained in $Q^*$ (that is, $\exists Q' \subset Q^*$ such that $\text{supp } a \subset Q'$, $\int a = 0$, $|a| \leq |Q'|^{-1}$).

Question. Can one state conditions on $V$ and $Q$ that guarantee $H^1_{\Delta - V} = H^1_Q$?
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If $Q_i, Q_j \in Q$ are neighbors, then diameters $d(Q_i)$ and $d(Q_j)$ are comparable.

Then we may take $\beta > 0$ small such that the family $\{Q^\beta\}_{j=1}^\infty$ has finite covering property, where $Q^\beta = (1 + \beta)Q$.

A function $a$ is an $H^1_Q$-atom if there exists $Q \in Q$ such that
- either $a = |Q|^{-1} 1_Q$
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Question. Can one state conditions on $V$ and $Q$ that guarantee $H^1_{\Delta-V} = H^1_Q$?
Examples of Hardy spaces for Schrödinger operators with local atoms.

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**Question.** Can one state conditions on \( V \) and \( Q \) that guarantee \( H^1_{\Delta-V} = H^1_Q \)?
Let $K_t(x, y)$ be the integral kernel of $e^{t(\Delta - V)}$.
We impose two assumptions on $V$ and $Q$ mainly: $(\exists C, \varepsilon, \delta > 0)$

$$\sup_{y \in Q^*} \int K_{2^n d(Q)^2} (x, y) \, dx \leq C n^{-1-\varepsilon} \quad \text{for } Q \in Q, n \in \mathbb{N}; \quad (D)$$

$$\int_0^{2t} (1_{Q^{***}} V) \ast h_s(x) \, ds \leq C \left( \frac{t}{d(Q)^2} \right)^{\delta} \quad (K)$$

for $x \in \mathbb{R}^d$, $Q \in Q$, $t \leq d(Q)^2$,

$$h_s(x) = (4\pi s)^{-d/2} e^{-|x|^2/(4s)}.$$

**Theorem. (J.D. and J. Zienkiewicz).**

Assume $(D)$ and $(K)$, then

$$f \in H_{\Delta - V}^1 \iff f \in H_Q^1 \quad \text{and} \quad \|f\|_{H_Q^1} \lesssim \|f\|_{H_{\Delta - V}^1} \lesssim \|f\|_{H_Q^1}.$$
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Meaning of the conditions (D) and (K).

The condition (D)

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Examples.

- $d \geq 3$, $V$ satisfies the reverse Hölder inequality with exponent $q > d/2$, that is,

$$\left(\frac{1}{|B|} \int_B V(y)^q \, dy\right)^{1/q} \leq C \frac{1}{|B|} \int_B V(y) \, dy$$

for every ball $B$.

Define $Q$ by: $Q \in \mathcal{Q}$ if and only if $Q$ is the maximal dyadic cube for which

$$d(Q)^2|Q|^{-1} \int_Q V(y) \, dy \leq 1.$$

Then the conditions (D) and (K) are true.

Every nonnegative polynomial satisfies the reverse Hölder inequality. The integral appeared in the article of C. Fefferman (joint work with D.H. Phong) about the uncertainty principle (Bull. AMS 1983). Then it was used by Z. Shen in his work about boundedness of the Riesz transforms $\partial_{x_j} L^{-1/2}$ on $L^p$, $p > 1$. 
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• The Hardy space $H^1_L$ associated with one-dimensional Schrödinger operator $-L$ was studied in Czaja-Zienkiewicz. For any nonnegative $V \in L^1_{loc}(\mathbb{R})$ the collection $Q$ of maximal dyadic intervals $Q$ of $\mathbb{R}$ that are defined by the stopping time condition

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fulfils (D) and (K) for certain small $\beta > 0$. 
\( V(x) = |x|^{-2}, \ d \geq 3. \) Then for \( Q \) being the Whitney decomposition of \( \mathbb{R}^d \setminus \{0\} \) that consists of dyadic cubes the conditions (D) and (K) hold.
More examples.

**Observation 1.** For \( \ell > 0 \) let \( Q_{\ell}(\mathbb{R}^n) \) be a covering of \( \mathbb{R}^n \) by cubes of length \( \ell \). Assume that \( V_1 \) na \( \mathbb{R}^d \) and \( Q \) conditions (D) i (K) hold.

Consider potential \( V(x_1, x_2) = V_1(x_1), \ x_1 \in \mathbb{R}^d, \ x_2 \in \mathbb{R}^n \), and family of cubes \( \tilde{Q} = \{ Q_1 \times Q_2 : Q_1 \in Q, \ Q_2 \in Q_d(Q_1)(\mathbb{R}^n) \} \) on \( \mathbb{R}^{d+n} \).

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Then the pair $(V, \tilde{Q})$ satisfies (D) i (K).
Proof of Observation 1 follows from the fact that the integral kernel for the semigroup generated by 
\( \mathcal{L} = \Delta - V_1(x_1) = \Delta_{\mathbb{R}^d} + \Delta_{\mathbb{R}^n} - V_1(x_1) \) is the product of integral kernels kernels of semigroups generated by \( L_1 = \Delta_{\mathbb{R}^d} - V_1(x_1) \) on \( \mathbb{R}^d \) and \( \Delta_{\mathbb{R}^n} \) on \( \mathbb{R}^n \);

\[ T_t((x_1, x_2); (y_1, y_2)) = T_1^1(x_1; y_1) h_t(x_2 - y_2), \]

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**Observation 2.** Let \( V_1, V_2 \) be nonnegative potentials on \( \mathbb{R}^d \), which together with families of cubes \( Q_1, Q_2 \) satisfy (D) i (K).

Assume additionally that \( Q_1, Q_2 \) consist of dyadic cubes. For \( Q_1 \in Q_1, Q_2 \in Q_2 \) the cubes are either disjoint or one contains another. Let \( Q_1 \cap Q_2 \) denotes the smaller one.

Then the family \( Q = \{ Q_1 \cap Q_2, Q_1 \in Q_1, Q_2 \in Q_2 \} \) covers \( \mathbb{R}^d \) and satisfy (D) i (K) for \( V = V_1 + V_2 \).

Proof. Let \( T_t(x, y), T_t^1(x, y), T_t^2(x, y) \) be the integral kernels for Schrödinger semigroups with potentials \( V, V_1, V_2 \) respectively. Then \( T_t(x, y) \leq \min(T_t^1(x, y), T_t^2(x, y)) \).

Assume that \( Q_1 = Q_1 \cap Q_2 \). Then (K) is trivial.

We check (D):

\[
\sup_{y \in Q_1^*} \int T_{2^n d(Q_1)}^2(x, y) \, dy \leq \sup_{y \in Q_1^*} \int T_{2^n d(Q_1)}^1(x, y) \, dy \leq C n^{-1-\varepsilon}.
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Observation 2. Let $V_1, V_2$ be nonnegative potentials on $\mathbb{R}^d$, which together with families of cubes $Q_1$ and $Q_2$ satisfy (D) i (K). Assume additionally that $Q_1$ and $Q_2$ consist of dyadic cubes.

For $Q_1 \in Q_1, Q_2 \in Q_2$ the cubes are either disjoint or one contains another. Let $Q_1 \wedge Q_2$ denotes the smaller one.

Then the family $Q = \{Q_1 \wedge Q_2, Q_1 \in Q_1, Q_2 \in Q_2\}$ covers $\mathbb{R}^d$ and satisfy (D) i (K) for $V = V_1 + V_2$.

Proof. Let $T_t(x,y), T_t^1(x,y), T_t^2(x,y)$ be the integral kernels for Schrödinger semigroups with potentials $V, V_1, V_2$ respectively.

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$$\sup_{y \in Q_1^*} \int T_{2^n d(Q_1)^2}(x, y) \, dy \leq \sup_{y \in Q_1^*} \int T^1_{2^n d(Q_1)^2}(x, y) \, dy \leq Cn^{-1-\varepsilon}.$$
Observation 2. Let $V_1, V_2$ be nonnegative potentials on $\mathbb{R}^d$, which together with families of cubes $Q_1$ i $Q_2$ satisfy (D) i (K). Assume additionally that $Q_1$ i $Q_2$ consist of dyadic cubes. For $Q_1 \in Q_1$, $Q_2 \in Q_2$ the cubes are either disjoint or one contains another. Let $Q_1 \wedge Q_2$ denotes the smaller one.

Then the family $Q = \{Q_1 \wedge Q_2, Q_1 \in Q_1, Q_2 \in Q_2\}$ covers $\mathbb{R}^d$ and satisfy (D) i (K) for $V = V_1 + V_2$.

Proof. Let $T_t(x, y), T_t^1(x, y), T_t^2(x, y)$ be the integral kernels for Schrödinger semigroups with potentials $V, V_1, V_2$ respectively.

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We check (D):

$$\sup_{y \in Q_1^*} \int T_{2^n d(Q_1)^2}(x, y) \, dy \leq \sup_{y \in Q_1^*} \int T_{2^n d(Q_1)^2}^1(x, y) \, dy \leq Cn^{-1-\varepsilon}.$$
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We check (D):

$$\sup_{y \in Q^*_1} \int T^{2n}d(Q_1)^2(x, y) \, dy \leq \sup_{y \in Q^*_1} \int T^{1n}d(Q_1)^2(x, y) \, dy \leq Cn^{-1-\varepsilon}.$$
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Then the family \( Q = \{ Q_1 \land Q_2, Q_1 \in Q_1, Q_2 \in Q_2 \} \) covers \( \mathbb{R}^d \) and satisfy (D) i (K) for \( V = V_1 + V_2 \).

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\]
Theorem (J.D. and M. Preisner)

In the case of (D) and (K) the space $H_{\Delta-V}^1$ is characterized by the Riesz transforms $R_j \partial x_j (-\Delta + V)^{-1/2}$, that is,

$$\|f\|_{H_{\Delta-V}^1} \sim \|f\|_{L^1} + \sum_{j=1}^{d} \|R_j f\|_{L^1}.$$
The smallest nontrivial non-negative potential is a compactly supported bounded function $V$.

Let $L = -\Delta + V$ in $\mathbb{R}^d$, $d \geq 3$, $V \in C_c(\mathbb{R}^d)$.

With J. Zienkiewicz we showed that there is a function $w$ such that $0 < \delta \leq w \leq 1$, $e^{t(\Delta - V)}w = w$ (w is $L$-harmonic) and the mapping

$$H_L^1 \ni f \mapsto wf \in H^1(\mathbb{R}^d)$$

is an isomorphism.

In other words: every atom satisfies the cancellation condition

$$\int a(x)w(x)\,dx = 0.$$ 

E. Stein asked question: For which potentials $V \geq 0$ on $\mathbb{R}^d$, $d \geq 3$ there is a function $w$ such that $0 < \delta \leq w \leq 1$ such that

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is an isomorphisms?
Assume that \( d \geq 3, \ V \geq 0, \ V \in L^1_{\text{loc}}(\mathbb{R}^d), \ K_t(x, y) \) kernels of \( e^{t(\Delta - V)} \).

**Theorem.** J.D. J. Zienkiewicz

The following are equivalent:

(a) There is an \( \Delta - V \)-harmonic function \( w: 0 < c \leq w(x) \leq C \);
(b) \( \int K_t(x, y) \, dy \geq \delta > 0 \)
(c) The global Kato norm

\[
\|V\|_K = \sup_{x \in \mathbb{R}^d} \int \frac{V(y)}{|x - y|^{d-2}} \, dy = \|\Delta^{-1}V\|_{L^\infty} \text{ is bounded;}
\]

(d) \( ct^{-d/2}e^{-C|x-y|^2/t} \leq K_t(x, y) \) (Gaussian lower bounds).
(e) there is a function \( 0 < \delta \leq w(x) \leq 1 \) such that
\( H^1_{\Delta - V} \ni f \mapsto w(x)f(x) \in H^1(\mathbb{R}^d) \) is an isomorphism.
Theorem (J.D. & J.Z.)

Assume that $\|V\|_K < \infty$, $d \geq 3$.

Then the operators

$$L^{1/2}(-\Delta)^{-1/2} \quad \text{and} \quad (-\Delta)^{1/2}L^{-1/2}$$

are bounded on $L^1$.

Moreover,

$$H^1_L \ni f \mapsto (-\Delta)^{1/2}L^{-1/2}f \in H^1(\mathbb{R}^d)$$

is isomorphism of the Hardy spaces with the inverse

$$H^1(\mathbb{R}^d) \ni f \mapsto L^{1/2}(-\Delta)^{-1/2}f \in H^1_L.$$
The Hardy space $H^1_L$ admits characterization by the Riesz transforms $R_j = \partial x_j L^{-1/2}$, that is, an $L^1$ function $f$ belongs to the Hardy space $H^1_L$ if and only if $R_j f \in L^1$.

Proof. Based on:

$$(\partial x_j (-\Delta)^{-1/2}) (\Delta)^{1/2} L^{-1/2} = \partial x_j L^{-1/2}$$

(classical Riesz transform) (isomorphism $H^1_L \to H^1(\mathbb{R}^d)$) (Riesz transform for $L$)

$$(\partial x_j L^{-1/2}) (L^{1/2} (-\Delta)^{-1/2}) = \partial x_j (-\Delta)^{-1/2}$$

(Riesz transform for $L$) (isomorphism $H^1(\mathbb{R}^d) \to H^1_L$) (classical Riesz transform)
Proof: $\partial x_j L^{-1/2} f \in L^1$ implies $f \in H^1_L$.

\[
\begin{align*}
\underbrace{(\partial x_j (-\Delta)^{-1/2})}_{\text{(classical Riesz transform)}} \quad \underbrace{(-\Delta)^{1/2} L^{-1/2}}_{\text{(isomorphism } H^1_L \to H^1(\mathbb{R}^d))} &= \underbrace{\partial x_j L^{-1/2}}_{\text{(Riesz transform for } L)}
\end{align*}
\]

Assume that
\[
\partial x_j L^{-1/2} f \in L^1 \quad j = 1, ..., d,
\]
then
\[
(-\Delta)^{1/2} L^{-1/2} f \in H^1(\mathbb{R}^d),
\]
so $f \in H^1_L$, because $(\partial x_j (-\Delta)^{-1/2})$ is an isomorphisms.
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$$(\partial_{x_j} (-\Delta)^{-1/2}) \quad (-\Delta)^{1/2} L^{-1/2} \quad = \quad \partial_{x_j} L^{-1/2}$$

(classical Riesz transform) (isomorphism $H^1_L \to H^1(\mathbb{R}^d)$) (Riesz transform for $L$)

Assume that

$$\partial_{x_j} L^{-1/2} f \in L^1 \quad j = 1, \ldots, d,$$

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\[
\begin{align*}
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The case of compactly supported potentials in dimension 2.

Assume that $V$ is a nonnegative nonzero compactly supported $C^2$-function in $\mathbb{R}^2$.

**Theorem (J.D. & J.Z.)**

There exists a regular $L$-harmonic weight $w$ such that

$$C^{-1} \ln(2 + |x|) \leq w(x) \leq C \ln(2 + |x|),$$

$$|\nabla w(x)| \leq C(1 + |x|)^{-1},$$

and the space $H_{L,max}^1$ admits atomic decomposition with atoms satisfying: $\text{supp } a \subset B$, $\|a\|_\infty \leq |B|^{-1}$, $\int a(x)w(x)\,dx = 0$. 

J. Dziubański (Wrocław)
Considered by many authors: B. Ahn, P. Auscher, F. Bernicot, S. Dekel, X.T. Duong, G. Garrigos, S. Hofmann, J. Huang, G. Kerkyacharian, G. Kyriazis, G. Liu, Y. Liu, T. Martinez, A. McIntosh, S. Mayboroda, D. Mitrea, M. Mitrea, P. Petrushev, M. Preisner, J. Torrea, L. Yan, Y. Xu, D. Yang, J. Zhao, ...
\((X, d(x, y)\mu)\) is a metric space with doubling Borel measure: 
\(\mu(B(x, 2r)) \leq C\mu(B(x, t))\) such that for very \(x \in X\) 
\((0, \infty) : t \mapsto \mu(B(x, t)) \in (0, \infty)\) is 1-1 and onto

\[ e^{-tL}f(x) = \int_X K_t(x, y)f(y)\,d\mu(y) \text{ semigroup on } L^2(\mu), L^* = L; \]

\[
\frac{c}{\mu(B(x, \sqrt{t}))} e^{-Cd(x,y)^2/t} \leq K_t(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-cd(x,y)^2/t}
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**Theorem, J.D. and M. Preisner:**

There is a function \(0 < \delta \leq w(x) \leq 1\) such that the Hardy space 
\(H^1_{L,\text{max}}\) admits atomic decomposition with atoms:

- \(\text{supp } a \subset B(y_0, r)\);
- \(\|a\|_{\infty} \leq \mu(B(y_0, r))^{-1}\)
- \(\int_X a(x)w(x)\,d\mu(x) = 0\).

We do not assume any regularity condition on \(K_t(x, y)\).
\((X, d(x, y)\mu)\) is a metric space with doubling Borel measure: 
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\[
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We do not assume any regularity condition on \(K_t(x, y)\)!
The Gaussian bounds ⇒ ∃ 0 < δ ≤ w(x) ≤ 1 and α > 0:

\[
\left| \frac{K_t(x, y)}{w(x)w(y)} - \frac{K_t(x, z)}{w(x)w(z)} \right| \lesssim \mu(B(x, \sqrt{t}))^{-1} \left( \frac{d(y, z)}{\sqrt{t}} \right)^\alpha \exp \left( -\frac{c d(x, y)^2}{t} \right)
\]

whenever \( d(y, z) \leq \sqrt{t} \) and \( \int_X K_t(x, y) \, d\mu(y) = w(x). \)

The proof uses spectral functional calculi for \( L \).

So, we may apply Uchiyama’s theorem about maximal function characterization of atomic Hardy spaces on spaces of homogeneous type by introducing the new kernel and space:

\[
T_t(x, y) = \frac{K_t(x, y)}{w(x)w(y)} \quad \text{on the space } (X, d(x, y), w(x)^2 d\mu(x)),
\]

\[
\int_X T_t(x, y)w(y)^2 \, d\mu(y) = 1.
\]
The Gaussian bounds \( \Rightarrow \exists 0 < \delta \leq w(x) \leq 1 \) and \( \alpha > 0 \):

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So,
\[ f \in H^1_{L,\text{max}} \iff \sup_{t > 0} \left| \int_X \frac{K_t(x,y)}{w(x)w(y)} \frac{f(y)}{w(y)} w(y)^2 \, d\mu(y) \right| \in \begin{cases} L^1(d\mu) \\ L^1(w(x)^2 \, d\mu) \end{cases} \]

\[
\frac{f(x)}{w(x)} \in H^1_{\text{atom}}(X, d, w(x)^2 \, d\mu(x))
\]

\[
\frac{f(x)}{w(x)} = \sum \lambda_j \tilde{a}_j(x)
\]

with \( \tilde{a}_j(x) \) atoms satisfying
\[
\int \tilde{a}_j(x) w(x)^2 \, d\mu(x) = 0
\]

\[
f(x) = \sum_j \lambda_j \tilde{a}_j(x) w(x) = \sum_j \lambda_j a_j(x), \quad a_j(x) = \tilde{a}_j(x) w(x), \]

\[
\int a_j(x) w(x) \, d\mu(x) = \int a_j(x) w(x) w(x) \, d\mu(x) = 0.
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\[ \left\{ L^1(w(x)^2 d\mu) \right\} \]

\[ \frac{f(x)}{w(x)} \in H^1_{\text{atom}}(X, d, w(x)^2 d\mu(x)) \]

\[ \frac{f(x)}{w(x)} = \sum \lambda_j \tilde{a}_j(x) \]

with \( \tilde{a}_j(x) \) atoms satisfying \( \int \tilde{a}_j(x) w(x)^2 \, d\mu(x) = 0 \)

\[ f(x) = \sum_j \lambda_j \tilde{a}_j(x) w(x) = \sum_j \lambda_j a_j(x), \quad a_j(x) = \tilde{a}_j(x) w(x), \]

\[ \int a_j(x) w(x) \, d\mu(x) = \int a_j(x) w(x) w(x) \, d\mu(x) = 0. \]
So,
\[ f \in H^1_{L,\max} \iff \sup_{t>0} \left| \int_X \frac{K_t(x,y)}{w(x)w(y)} \frac{f(y)}{w(y)} w(y)^2 \, d\mu(y) \right| \in L^1(d\mu) \cup L^1(w(x)^2 \, d\mu) \]

\[ \frac{f(x)}{w(x)} \in H^1_{\text{atom}}(X, d, w(x)^2 \, d\mu(x)) \]

\[ \frac{f(x)}{w(x)} = \sum \lambda_j \tilde{a}_j(x) \]

with \( \tilde{a}_j(x) \) atoms satisfying \( \int \tilde{a}_j(x)w(x)^2 \, d\mu(x) = 0 \).

\[ f(x) = \sum_{j} \lambda_j \tilde{a}_j(x)w(x) = \sum_{j} \lambda_j a_j(x), \quad a_j(x) = \tilde{a}_j(x)w(x), \]

\[ \int a_j(x)w(x) \, d\mu(x) = \int a_j(x)w(x)w(x) \, d\mu(x) = 0. \]
So, \( f \in H^1_{L,\max} \iff \sup_{t>0} \left| \int_X \frac{K_t(x,y)}{w(x)w(y)} \frac{f(y)}{w(y)} w(y)^2 \, d\mu(y) \right| \in L^1(d\mu) \subseteq L^1(w(x)^2d\mu) \)

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Grushin operator

\[ L = -\Delta_x - |x|^2 \frac{\partial^2}{\partial x'\partial x'^2}, \quad x = (x, x') = (x_1, \ldots, x_n, x') \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \]

\[ \rho(x, y) \] control distance for \( \partial_{x_1}, \ldots, \partial_{x_n}, x_1 \partial_{x'}, \ldots, x_n \partial_{x'} \)

\((\mathbb{R}^{n+1}, \rho(x, y), dx)\) space of homogeneous type

\[ M_L f(x) = \sup_{t > 0} |e^{-tL} f(x)| \] max function for the heat semigroup

\[ H^1_L = \{ f \in L^1 : \| f \|_{H^1_L} = \| M_L f \|_{L^1} < \infty \} \]

Riesz transforms: \( R_j = \partial_{x_j} L^{-1/2}, \quad R_{j+n} = x_j \partial_{x'} L^{-1/2} \), \( j = 1, \ldots, n \).

**Theorem (J.D. and K. Jotsaroop).**

The space \( H^1_L \) admits atomic decomposition with atoms for the space of homogeneous type.

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Two proofs of the Riesz transform characterization of $H^1$

- subharmonicity + maximum principle for subharmonic functions
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Riesz transform characterization implies the Fefferman-Stein decomposition of $BMO$ functions:

every $g \in VMO$ can be written $g = g_0 + \sum_{j=1}^{d} R_j g_j$
with $\sum_{j=0}^{d} \|g_j\|_\infty \leq C\|g\|_{BM0}$

Uchiyama did opposite: proved the Fefferman-Stein decomposition of $VMO$ functions and then concluded the Riesz transform characterization of $H^1(\mathbb{R}^d)$. Actually he characterized systems $(R_0, R_1, ..., R_m)$ of singular integrals which characterize $H^1(\mathbb{R}^d)$.

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First we prove Fefferman-Stein decomposition of compactly supported $BMO(\mathbb{R}^{n+1}, \rho, d\mathbf{x})$ functions:

$$g = g_0 + \sum_{k=1}^{2n} R_k g_k, \quad \text{with} \quad \sum_{k=0}^{2n} \|g_k\|_\infty < C \|g\|_{BMO}, \ g_j \in L^2$$

To this end we relate $L$ with $\Delta = -\sum_{j=1}^{n}(X_j^2 + Y_j^2)$ on the Heisenberg group $\mathbb{H}_n$ via the unitary representation

$$\pi_{(a,b,t)} f(x) = \pi_{(a,b,t)} f(x, x') = f(x + b, x' + t + \frac{1}{2}a \cdot b + a \cdot x),$$

$$\pi_\Delta = L, \ \pi_{X_j} = x_j \partial_{x'}, \ \pi_{Y_j} = \partial_{x_j}, \ \pi_{X_j(\Delta)^{-1/2}} = x_j \partial_{x'} L^{-1/2}, \ldots$$

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Hardy spaces in the Dunkl setting

Dunkl = deformation of Fourier

Parameter: \( k \geq 0 \)

\[ k = 0 \] \( \mathbb{R}^N \ni x \) Euclidean space, for \( 0 \neq \alpha \in \mathbb{R}^N \),

\[ \sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha \] reflection in the hyperplane \( \alpha^\perp \)

\( R \) root system: finite set of vectors \( \alpha \neq 0 \): \( \sigma_\alpha(R) = R \), \( \forall \alpha \in R \)

\( G \) (Coxeter) group generated by reflections \( \sigma_\alpha, \alpha \in R \).
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J. Dziubański (Wrocław)  Function Spaces and their Applications
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Assumptions: \( R \) is reduced: \( R \cap \mathbb{R} \alpha = \{-\alpha, \alpha\} \)

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Dunkl operator (Dunkl derivative)

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D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} \left\{ f(x) - f(\sigma_\alpha x) \right\}
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Dunkl operator in dimension one and in the product case

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Df(x) = f'(x) + \frac{k}{x} \left\{ f(x) - f(-x) \right\}
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Skewsymmetry:
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measure: \( dw_k(x) = w_k(x) \, dx \), \( w_k(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)} \);

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Dunkl Laplacian \( \Delta_k \): fix an orthonormal basis \( e_1, \ldots, e_N \) of \( \mathbb{R}^N \);

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\Delta_k f(x) = \sum_{j=1}^{N} D_{e_j}^2 f(x)
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Heat and Poisson semigroups

\[ e^{t\Delta_k} f(x) = \int_{\mathbb{R}^N} h_t(x,y) f(y) \, dw_k(y), \]

\( h_t(x,y) \) is not Gaussian!

\[ e^{-t\sqrt{-\Delta_k}} f(x) = \int_{\mathbb{R}^N} P_t(x,y) f(y) \, dw_k(y), \]

\( P_t f \) solves the initial (boundary value problem):

\((\partial_t^2 + \Delta_k)u(t,x) = 0, \ u(0,x) = f(x). \)

Knowledge about the kernels \( h_t(x,y) \) and \( P_t(x,y) \) was rather poor for our purposes, so our first task was to derive estimates for them.

Purpose: study \( \mathcal{L} = (\partial_t^2 + \Delta_k) \)-harmonic functions in the half-space \((0, \infty) \times \mathbb{R}^N\) and conjugate harmonic functions.

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Unify notation: \( x_0 = t, \ D_0 = \partial_{e_0}, \ L = \partial_{e_0}^2 + \Delta_k = \sum_{j=0}^N D_j^2. \)

\[ u = (u_0, u_1, \ldots, u_N), \text{ satisfies Cauchy-Riemann equations if} \]

\[ D_j u_\ell = D_\ell u_j, \quad \sum_{j=0}^N D_j u_j = 0, \quad \text{(CR)} \]

Then \( \Delta_k u_j = 0. \)

\[ \|u\|_{\mathcal{H}_k^1} = \sup_{x_0 > 0} \int_{\mathbb{R}^N} |u(x_0, x_1, \ldots, x_N)| \, dw_k(x_1, \ldots, x_N) < \infty. \quad (L^1) \]
Unify notation: \( x_0 = t, \ D_0 = \partial e_0, \ L = \partial^2 e_0 + \Delta_k = \sum_{j=0}^{N} D_j^2. \)

\( \mathbf{u} = (u_0, u_1, \ldots, u_N), \) satisfies Cauchy-Riemann equations if

\[
D_j u_\ell = D_\ell u_j, \quad \sum_{j=0}^{N} D_j u_j = 0, \quad \text{(CR)}
\]

Then \( \Delta_k u_j = 0. \)

\[
\| \mathbf{u} \|_{\mathcal{H}_k^1} = \sup_{x_0 > 0} \int_{\mathbb{R}^N} |\mathbf{u}(x_0, x_1, \ldots, x_N)| \, dw_k(x_1, \ldots, x_N) < \infty. \quad (L^1)
\]
Relation with Riesz transforms:

\[ R_j f(x) = -D_j (-\Delta_k)^{-1/2} f(x). \]

\[ (P_{x_0} f(x), R_1 P_{x_0} f(x), \ldots, R_N P_{x_0} f(x)) \] satisfies (CR)

Our results are:

**Theorem 1.**

Let \( u_0 \) be \( \mathcal{L} \)-harmonic. Then there is a vector \( u = (u_0, u_1, \ldots, u_N) \) satisfying (CR) and \( (L^1) \) if and only if

\[ u_0^*(x) = \sup_{|x' - x| < x_0} |u_0(x_0, x)| \]

belongs to \( L^1(dw_k) \), \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) and

\[ \|u\|_{\mathcal{H}^1_k} \sim \|u^*\|_{L^1(dw_k)}. \]
Relation with Riesz transforms:

\[ R_j f(x) = -D_j ( -\Delta_k )^{-1/2} f(x). \]

\((P_{x_0} f(x), R_1 P_{x_0} f(x), ..., R_N P_{x_0} f(x))\) satisfies \((\text{CR})\)

**Our results are:**

**Theorem 1.**

Let \( u_0 \) be \( L \)-harmonic. Then there is a vector \( u = (u_0, u_1, ..., u_N) \) satisfying (CR) and \((L^1)\) if and only if

\[ u^*_0(x) = \sup_{|x' - x| < x_0} |u_0(x_0, x)| \]

belongs to \( L^1(dw_k) \), \( x = (x_1, ..., x_N) \in \mathbb{R}^N \) and

\[ \| u \|_{\mathcal{H}^1_k} \sim \| u^* \|_{L^1(dw_k)}. \]
Real Hardy space $H_{\Delta_k}^1$

If $(u_0, u_1, \ldots, u_N) \in \mathcal{H}_{k}^1$, then $\lim_{x_0 \to 0} u_0(x_0, x)$ exists in $L^1(dw_k)$.

$$H_{\Delta_k}^1(\mathbb{R}^N) = \{ f(x) = \lim_{x_0 \to 0} u_0(x_0, x) : (u_0, u_1, \ldots, u_N) \in \mathcal{H}_{k}^1 \}$$

Theorem 2. Characterizations of $H_{\Delta_k}^1$ by

- Riesz transforms: $f \in H_{\Delta_k}^1$ iff $R_j f \in L^1(dw_k)$;

- square functions:
  $$Sf(x) = \left( \iint_{|x-y|<t} t^2 \Delta_k e^{t^2 \Delta_k} f(y) \frac{dw_k(y)dt}{w_k(B(x,t))t} \right)^{1/2}$$

- other maximal functions (heat semigroup)

- atomic decompositions
Thank you