Morrey spaces and A_p weights: embeddings and extrapolation

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Morrey spaces (unweighted)

Let $0 < p < \infty$ and $-\frac{n}{p} \le r < 0$: $L^r_p(\mathbb{R}^n)$ is the space of functions such that

$$\left\| f | L^r_p \right\| \equiv \sup_Q \frac{1}{|Q|^{\frac{1}{p}+\frac{r}{n}}} \left(\int_Q |f|^p \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Another notation: $L_{p,\lambda}$ with $0 \le \lambda < n$ and

$$\left\| f | L_{p,\lambda} \right\| \equiv \sup_Q \left(\frac{1}{\ell(Q)^\lambda} \int_Q |f|^p \right)^{\frac{1}{p}},$$

$$\lambda = n + rp$$

$L^{-n/p}_p(\mathbb{R}^n) = L_p$

Weighted Morrey spaces

Let $0 < p < \infty$ and $-\frac{n}{p} \leq r < 0$. A weight w is a nonnegative locally integrable function in $\mathbb{R}^n$. We consider two types of weighted Morrey spaces:

- $$\left\| f | L_p^r(w) \right\| \equiv \sup_Q \frac{1}{w(Q)^{\frac{1}{p}+\frac{r}{n}}} \left(\int_Q |f|^p w \right)^{\frac{1}{p}}.$$
- $$\left\| f | L_p^r(\lambda, w) \right\| \equiv \sup_Q \frac{1}{|Q|^{\frac{1}{p}+\frac{r}{n}}} \left(\int_Q |f|^p w \right)^{\frac{1}{p}}.$$

(Both are $L_p^r(\mathbb{R}^n)$ for $w \equiv 1$.)

$$L_p^{-n/p}(w) = L_p^{-n/p}(\lambda, w) = L_{p,w} \ \left(:= \{ f : \textstyle\int_{\mathbb{R}^n} |f|^p w < \infty \} \right)$$

A_p weights

$1 < p < \infty$: $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty.$$

$p = 1$: $w \in A_1$ if

$$\frac{w(Q)}{|Q|} \leq C \inf_{x \in Q} w(x).$$

- $A_1 \subset A_p \subset A_q$ $(1 < p < q)$.
- $1 < p < \infty$: M (Hardy-Littlewood maximal operator) is bounded on $L^p(w)$ if and only if $w \in A_p$.

Reverse Hölder inequality

w is in the *reverse Hölder class RH_σ* for $1 < \sigma < \infty$ if

$$\left(\frac{1}{|Q|}\int_Q w(x)^\sigma dx\right)^{\frac{1}{\sigma}} \leq \frac{C}{|Q|}\int_Q w(x)dx,$$

with C independent of Q.

If $w \in RH_\sigma$ and $E \subset Q$, then $\dfrac{w(E)}{w(Q)} \leq C\left(\dfrac{|E|}{|Q|}\right)^{1/\sigma'}$.

- If $w \in A_p$, then $w \in RH_\sigma$ for some $\sigma > 1$.
- $A_p \cap RH_\sigma = \{w : w^\sigma \in A_{\sigma(p-1)+1}\}$.

A typical example: power weights

Let $w(x) = |x|^\alpha$.

- $w \in A_p$ if and only if $-n < \alpha < n(p-1)$.
- For $\alpha \geq 0$, $w \in RH_\sigma$ for every σ.
- For $-n < \alpha < 0$, $w \in RH_\sigma$ for $\sigma < \frac{n}{-\alpha}$.

Operators on Morrey spaces

Smooth functions are not dense in the Morrey spaces.

We cannot use a density argument to extend an operator defined in the class of smooth functions and satisfying estimates with Morrey norms. We still need a way to define/extend the operator to the full Morrey space.

The conclusion of our work can be summarize as follows:

> *If an operator satisfies (enough) weighted inequalities with A_p weights, it is already defined on the corresponding Morrey space and it satisfies Morrey estimates.*

Embeddings: weighted Morrey into weighted Lebesgue

Proposition

- Let $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$ and $w \in A_p$. For q_0 near 1 and all $q \in [1, q_0]$ there is $0 < \alpha < n$ such that

$$L_p^r(w) \hookrightarrow L_{q,(1+|x|)^{-\alpha}}.$$

- If moreover $w \in RH_\sigma$ and $-\frac{n}{p} \leq r \leq -\frac{n}{p\sigma}$, it also holds that

$$L_p^r(\lambda, w) \hookrightarrow L_{q,(1+|x|)^{-\alpha}}.$$

In particular $r = -\frac{n}{p}$ gives $L_{p,w} \hookrightarrow L_{q,(1+|x|)^{-\alpha}}$.

Embeddings: weighted Morrey into weighted Lebesgue

Similar result for endpoints:

Proposition

Let $1 \leq p < \infty$ and $\nu \geq 1$. Let $w \in A_1 \cap RH_\sigma$ for some $\sigma > \nu$. Then for r in an appropriate range, there exists some $0 < \alpha < n/\nu$ such that

$$L_p^r(w), L_p^r(\lambda, w) \hookrightarrow L_{p,(1+|x|)^{-\alpha}}(\mathbb{R}^n).$$

A property of weighted Lebesgue spaces

Proposition

Let be $1 < q < \infty$. Then it holds that

$$\bigcup_{w \in A_q} L_{q,w} = \bigcup_{1<p<\infty} \bigcup_{w \in A_p} L_{p,w} \subsetneq \bigcup_{w \in A_1} L_{1,w}.$$

Hence the left-hand side is independent of q.

This result is also contained in G. Knese, J. E. McCarthy, and K. Moen: Unions of Lebesgue spaces and A_1 majorants. *Pacific J. Math.* 280 (2016), no. 2, 411–432.

Weighted Morrey contained in the union of weighted Lebesgue spaces

Corollary

For every $1 \le p_0 < \infty$, every $1 < p < \infty$ (and also $p = 1$ if $p_0 = 1$), $-\frac{n}{p} \le r < 0$ and $w \in A_p$, it holds that

$$L_p^r(w) \subset \bigcup_{u \in A_{p_0}} L_{p_0,u}.$$

If moreover $w \in A_p \cap RH_\sigma$, for every $-\frac{n}{p} \le r < -\frac{n}{p\sigma}$ it holds that

$$L_p^r(\lambda, w) \subset \bigcup_{u \in A_{p_0}} L_{p_0,u}.$$

Extrapolation of weighted inequalities (Lebesgue spaces)

J. L. Rubio de Francia, 1982

Let $1 \leq p_0 < \infty$. If a sublinear operator is bounded on $L_{p_0,w}$ for all $w \in A_{p_0}$, it is bounded on $L_{p,w}$ for all $w \in A_p$ and $1 < p < \infty$.

Several extensions and variants. In particular, operators are not needed.

Let $1 \leq p_0 < \infty$. If for pairs (f, g) in a collection $\mathcal{F}$,

$$\|g|L_{p_0,w}\| \leq C\,\|f|L_{p_0,w}\|\,,$$

for all $w \in A_{p_0}$, then the inequality holds for the same pairs with $L_{p,w}$-norms for all $w \in A_p$ and $1 < p < \infty$.

Extrapolation-type results (Lebesgue to Morrey)

Theorem

Let $1 \leq p_0 < \infty$ and let $\mathcal{F}$ be a collection of nonnegative measurable pairs of functions. Assume that for every $(f, g) \in \mathcal{F}$ and every $w \in A_{p_0}$ we have

$$\|g|L_{p_0,w}\| \leq c_1 \|f|L_{p_0,w}\|,$$

where c_1 does not depend on the pair (f, g) and it depends on w only in terms of $[w]_{A_{p_0}}$. Then for every $1 < p < \infty$ (and also for $p = 1$ if $p_0 = 1$), every $-\frac{n}{p} \leq r < 0$ and every $w \in A_p$ we have

$$\left\|g|L_p^r(w)\right\| \leq c_2 \left\|f|L_p^r(w)\right\|.$$

Furthermore, for every $1 < p < \infty$ (and also for $p = 1$ if $p_0 = 1$) and $w \in A_p \cap RH_\sigma$, if $-\frac{n}{p} \leq r \leq -\frac{n}{p\sigma}$ we have

$$\left\|g|L_p^r(\lambda, w)\right\| \leq c_3 \left\|f|L_p^r(\lambda, w)\right\|.$$

Comments on the statement

$$\|g|L_{p_0,w}\| \leq c_1 \|f|L_{p_0,w}\| \quad \text{for all } w \in A_{p_0} \qquad (*)$$

- If (*) is assumed to hold whenever the right-hand side is finite (that is, LHS $= \infty \Rightarrow$ RHS $= \infty$), the theorem asserts that it f is in $L^r_p(w)$ (resp. $L^r_p(\lambda, w)$), then also g is in $L^r_p(w)$ (resp. $L^r_p(\lambda, w)$).
- If (*) is only assumed to hold whenever the left-hand side is finite ($\infty \leq$ finite is not excluded), the conclusion holds only for those g which are in $L^r_p(w)$ (resp. $L^r_p(\lambda, w)$).

Details of the proof

$w \in A_p$, $q > 1$ such that $w \in A_{p/q}$. Set $\tilde{p} = p/q$. Fix a cube Q. By duality

$$\left(\int_Q g^p w\right)^{\frac{1}{p}} = \left(\int_Q g^{\tilde{p}q} w\right)^{\frac{1}{\tilde{p}q}} = \sup_{h:\,\|h|L_{\tilde{p}',w}(Q)\|=1} \left(\int_Q g^q h w\right)^{\frac{1}{q}}.$$

Fix h and we have

$$\left(\int_{\mathbb{R}^n} g^q h w \chi_Q\right)^{\frac{1}{q}} \overset{\dagger}{\leq} \left(\int_{\mathbb{R}^n} g^q M(h^s w^s \chi_Q)^{\frac{1}{s}}\right)^{\frac{1}{q}} \overset{\dagger\dagger}{\leq} c \left(\int_{\mathbb{R}^n} f^q M(h^s w^s \chi_Q)^{\frac{1}{s}}\right)^{\frac{1}{q}}$$

$\dagger$: $h^s w^s \chi_Q \leq M(h^s w^s \chi_Q)$;
$\dagger\dagger$: $M(h^s w^s \chi_Q)^{1/s} \in A_1 \subset A_q$ for $s > 1$ and hypothesis.

Details of the proof

$$\left(\int_{2Q} f^q M(h^s w^s \chi_Q)^{\frac{1}{s}}\right)^{\frac{1}{q}} \leq \left(\int_{2Q} f^p w\right)^{\frac{1}{p}} \left(\int_{2Q} M(h^s w^s \chi_Q)^{\frac{\tilde{p}'}{s}} w^{1-\tilde{p}'}\right)^{\frac{1}{q\tilde{p}'}} .$$

- $f \in L_p^r(w)$
- boundedness of M.

$$\int_{\mathbb{R}^n \setminus 2Q} f^q M(h^s w^s \chi_Q)^{\frac{1}{s}} \leq c_1 \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} f^q \left(\frac{\int_Q h^s w^s}{|2^j Q|}\right)^{\frac{1}{s}}$$

- Hölder to $\int_{2^{j+1}Q} f^p w$ and $f \in L_p^r(w)$,
- $\left(\int_Q h^s w^s\right)^{1/s} \leq c\, w(Q)^{1/\tilde{p}} |Q|^{-1/s'}$,
- properties of $w \in A_p$.

Application to operators

Corollary

Let $1 \leq p_0 < \infty$. T acts from $\bigcup_{w \in A_{p_0}} L_{p_0,w}$ into the space of measurable functions and

$$\| Tf | \, L_{p_0,w} \| \leq c_1 \, \| f | \, L_{p_0,w} \|$$

for all $f \in L_{p_0,w}$ and $w \in A_{p_0}$, with a constant depending on $[w]_{A_{p_0}}$. Then for every $1 < p < \infty$ (and also $p = 1$ if $p_0 = 1$), every $-\frac{n}{p} \leq r < 0$ and every $w \in A_p$, we have that T is well defined on $L_p^r(w)$ by restriction and, moreover,

$$\| Tf | \, L_p^r(w) \| \leq c_2 \, \| f | \, L_p^r(w) \| \, ,$$

for every $f \in L_p^r(w)$.

Application to operators

Furthermore, for p as before and every $w \in A_p \cap RH_\sigma$, if $-\frac{n}{p} \leq r \leq -\frac{n}{p\sigma}$ we have that T is well defined on $L_p^r(\lambda, w)$ by restriction and, moreover,

$$\| Tf | L_p^r(\lambda, w)\| \leq c_3 \| f | L_p^r(\lambda, w)\|$$

for every $f \in L_p^r(\lambda, w)$.

If T satisfies the weak-type assumption

$$\| Tf | WL_{p_0,w}\| \leq c_1 \| f | L_{p_0,w}\|$$

the estimates are replaced by their weak-type counterparts (Morrey spaces defined with a weak-type norm).

Other consequences

Vector-valued estimates: for $1 < q, p < \infty$ and same conditions on r,

$$\left\| \Big(\sum_j |T_j f_j|^q \Big)^{1/q} \Big| L_p^r(w) \right\| \leq c_4 \left\| \Big(\sum_j |f_j|^q \Big)^{1/q} \Big| L_p^r(w) \right\|,$$

Similar estimates on $L_p^r(\lambda, w)$.

Extensions

- Hypothesis of the type: for $1 < \gamma \leq p_0 < \infty$ and every $w \in A_{p_0/\gamma}$,

$$\left\| g | L_{p_0,w} \right\| \leq c_1 \left\| f | L_{p_0,w} \right\| .$$

  To be applied to operators like rough singular integrals, Mihlin-Hörmander multipliers, etc.

- *(Limited range extrapolation)* For some p_0 such that $1 \leq p_- \leq p_0 \leq p_+ < \infty$ and $w \in A_{p_0/p_-} \cap RH_{(p_+/p_0)'}$ we have

$$\left\| g | \, L_{p_0,w}(\mathbb{R}^n) \right\| \leq c_1 \left\| f | \, L_{p_0,w}(\mathbb{R}^n) \right\| ,$$

  To be applied to some operators, like Bochner-Riesz multipliers and others, bounded on L^p for a limited range of p's, with enough weighted estimates.

Applications

Directly applied to many operators.

- Calderón-Zygmund operators and their maximal truncations
- Multipliers
- Rough singular integrals
- Square functions of different types
- Commutators
- Oscillatory singular integrals
- Bochner-Riesz operators
- Pseudodifferential operators

A_∞-type results

Let $0 < p < \infty$ and $w \in A_\infty$. Then the following inequalities hold whenever the left-hand side is finite:

$$\left\| Tf \middle| L_p^r(w) \right\| \le c_1 \left\| Mf \middle| L_p^r(w) \right\|,$$
$$\left\| Mf \middle| L_p^r(w) \right\| \le c_2 \left\| M^\sharp f \middle| L_p^r(w) \right\|,$$
$$\left\| I_\alpha f \middle| L_p^r(w) \right\| \le c_3 \left\| M_\alpha f \middle| L_p^r(w) \right\|.$$

T: Calderón-Zygmund operator,
M: Hardy-Littlewood maximal operator,
$M^\sharp$: the sharp maximal function,
I_α: fractional integral of order $\alpha \in (0, n)$,
M_α: fractional maximal operator of order α.

Also with $L_p^r(\lambda, w)$ instead of $L_p^r(w)$ if $w \in RH_\sigma$ and $-\frac{n}{p} \le r \le -\frac{n}{p\sigma}$.

Reference

Joint work with Marcel Rosenthal

Extension and boundedness of operators on Morrey spaces from extrapolation techniques and embeddings

`arxiv.org/abs/1607.04565` (v2)