Entropy and approximation numbers of embeddings of spaces of block-radial functions.

Alicja Dota*, Leszek Skrzypczak**

Poznań University of Technology,Institute of Mathematics*
Adam Mickiewicz University, Faculty of Mathematics and Computer Science**

Będlewo 2017

Table of contents

1. Spaces of block-radial functions
2. Entropy and approximation numbers
3. The main results
4. Applicationes in spceral theorem

Block-radial symmetry

- Let $m \in \{1, \ldots, d\}$ and let $\gamma \in \mathbb{N}^m$ be an m-tuple $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 + \ldots + \gamma_m = |\gamma| = d$. The m-tuple γ describes decomposition of

$$\mathbb{R}^d = \mathbb{R}^{|\gamma|} = \mathbb{R}^{\gamma_1} \times \cdots \times \mathbb{R}^{\gamma_m}$$

  into m subspaces of dimensions $\gamma_1, \ldots, \gamma_m$ respectively.
- Let

$$SO(\gamma) = SO(\gamma_1) \times \ldots \times SO(\gamma_m) \subset SO(d)$$

  be a group of isometries on $\mathbb{R}^{|\gamma|}$.
  An element $g = (g_1, \ldots, g_m)$, $g_i \in SO(\gamma_i)$ acts on $x = (\tilde{x}_1, \ldots, \tilde{x}_m)$, $\tilde{x}_i \in \mathbb{R}^{\gamma_i}$ by $x \mapsto g(x) = (g_1(\tilde{x}_1), \ldots, g_m(\tilde{x}_m))$.
- If $m = 1$ then $SO(\gamma) = SO(d)$ is a special orthogonal group acting on $\mathbb{R}^d$. If $m = d$ then the group is trivial since then $\gamma_1 = \ldots = \gamma_m = 1$ and $SO(1) = \{\mathrm{id}\}$. We will always assume that $\gamma_i \geq 2$ for any $i = 1, \ldots, m$.

Block-radial symmetry

- Let $m \in \{1, \ldots, d\}$ and let $\gamma \in \mathbb{N}^m$ be an m-tuple $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 + \ldots + \gamma_m = |\gamma| = d$. The m-tuple γ describes decomposition of

$$\mathbb{R}^d = \mathbb{R}^{|\gamma|} = \mathbb{R}^{\gamma_1} \times \cdots \times \mathbb{R}^{\gamma_m}$$

  into m subspaces of dimensions $\gamma_1, \ldots, \gamma_m$ respectively.
- Let

$$SO(\gamma) = SO(\gamma_1) \times \ldots \times SO(\gamma_m) \subset SO(d)$$

  be a group of isometries on $\mathbb{R}^{|\gamma|}$.
  An element $g = (g_1, \ldots, g_m)$, $g_i \in SO(\gamma_i)$ acts on $x = (\tilde{x}_1, \ldots, \tilde{x}_m)$, $\tilde{x}_i \in \mathbb{R}^{\gamma_i}$ by $x \mapsto g(x) = (g_1(\tilde{x}_1), \ldots, g_m(\tilde{x}_m))$.
- If $m = 1$ then $SO(\gamma) = SO(d)$ is a special orthogonal group acting on $\mathbb{R}^d$. If $m = d$ then the group is trivial since then $\gamma_1 = \ldots = \gamma_m = 1$ and $SO(1) = \{\mathrm{id}\}$. We will always assume that $\gamma_i \geq 2$ for any $i = 1, \ldots, m$.

Block-radial symmetry

- Let $m \in \{1, \ldots, d\}$ and let $\gamma \in \mathbb{N}^m$ be an m-tuple $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 + \ldots + \gamma_m = |\gamma| = d$. The m-tuple γ describes decomposition of

$$\mathbb{R}^d = \mathbb{R}^{|\gamma|} = \mathbb{R}^{\gamma_1} \times \cdots \times \mathbb{R}^{\gamma_m}$$

  into m subspaces of dimensions $\gamma_1, \ldots, \gamma_m$ respectively.
- Let

$$SO(\gamma) = SO(\gamma_1) \times \ldots \times SO(\gamma_m) \subset SO(d)$$

  be a group of isometries on $\mathbb{R}^{|\gamma|}$.
  An element $g = (g_1, \ldots, g_m)$, $g_i \in SO(\gamma_i)$ acts on $x = (\tilde{x}_1, \ldots, \tilde{x}_m)$, $\tilde{x}_i \in \mathbb{R}^{\gamma_i}$ by $x \mapsto g(x) = (g_1(\tilde{x}_1), \ldots, g_m(\tilde{x}_m))$.
- If $m = 1$ then $SO(\gamma) = SO(d)$ is a special orthogonal group acting on $\mathbb{R}^d$. If $m = d$ then the group is trivial since then $\gamma_1 = \ldots = \gamma_m = 1$ and $SO(1) = \{\mathrm{id}\}$. We will always assume that $\gamma_i \geq 2$ for any $i = 1, \ldots, m$.

Block-radial Besov spaces

Definition

Let $s \in \mathbb{R}$ and $1 < p, q \leq \infty$. Then

$$B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \left\| f | B^s_{p,q} \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right\|_p^q \right)^{1/q} < \infty \right\}$$

is called Besov space. Here $\{\varphi_j\}$ is a smooth dyadic partition of unity.

- By $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ we mean the subset of $SO(\gamma)$-invariant distributions in $B^s_{p,q}(\mathbb{R}^d)$ and we endow this subspace with the same norm as the orginal space.
- If $SO(\gamma) = SO(d)$ then the subspace consists of radial functions

Block-radial Besov spaces

Definition

Let $s \in \mathbb{R}$ and $1 < p, q \leq \infty$. Then

$$B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \left\| f | B^s_{p,q} \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right\|_p^q \right)^{1/q} < \infty \right\}$$

is called Besov space. Here $\{\varphi_j\}$ is a smooth dyadic partition of unity.

- By $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ we mean the subset of $SO(\gamma)$-invariant distributions in $B^s_{p,q}(\mathbb{R}^d)$ and we endow this subspace with the same norm as the orginal space.
- If $SO(\gamma) = SO(d)$ then the subspace consists of radial functions

Block-radial Besov spaces

Definition

Let $s \in \mathbb{R}$ and $1 < p, q \leq \infty$. Then

$$B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \left\| f | B^s_{p,q} \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right\|_p^q \right)^{1/q} < \infty \right\}$$

is called Besov space. Here $\{\varphi_j\}$ is a smooth dyadic partition of unity.

- By $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ we mean the subset of $SO(\gamma)$-invariant distributions in $B^s_{p,q}(\mathbb{R}^d)$ and we endow this subspace with the same norm as the orginal space.
- If $SO(\gamma) = SO(d)$ then the subspace consists of radial functions

- (L.Skrzypczak 2002) It is known that the embading

$$id \ : \ R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d) \tag{1}$$

is compact if and only if

$$p_1 < p_2, \quad \delta = s_1 - \frac{d}{p_1} - s_2 + \frac{d}{p_2} > 0 \quad \text{and} \quad \min_i \gamma_i \geq 2\,. \tag{2}$$

Entropy and approximation numbers

Definition

Let X and Y be Banach spaces and $T \in L(X, Y)$ be a linear operator. The k-th entropy numbers of T, $k \in \mathbb{N}$, is defined in the following way

$$e_k(T) := \inf\{\epsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \epsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X.

The sequence of entropy numbers can be viewed as quantification of the notion of compactness since the operator is compact if and only if $e_k(T) \to 0$, as $k \to \infty$.

Definition

Let X and Y be Banach spaces and $T \in L(X, Y)$ be a linear operator. The k-th entropy numbers of T, $k \in \mathbb{N}$, is defined in the following way

$$e_k(T) := \inf\{\epsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \epsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X.

The sequence of entropy numbers can be viewed as quantification of the notion of compactness since the operator is compact if and only if $e_k(T) \to 0$, as $k \to \infty$.

Definition

Let X and Y be Banach spaces and $T \in L(X, Y)$ be a linear operator. The k-th approximation numbers of T is defined as

$$a_k(T) := \inf\{\|T - A\| \ : \ A \in L(X, Y), \ \mathrm{rank}(A) < k\},$$

where rank(A) denotes dimension of the range $A(X) = \{A(x), x \in X\}$.

Now $a_k(T) \to 0$ implies that T is compact. The converse implication generally does not hold.

Definition

Let X and Y be Banach spaces and $T \in L(X,Y)$ be a linear operator. The k-th approximation numbers of T is defined as

$$a_k(T) := \inf\{\|T - A\| \ : \ A \in L(X,Y), \ \mathrm{rank}(A) < k\},$$

where $\mathrm{rank}(A)$ denotes dimension of the range $A(X) = \{A(x), x \in X\}$.

Now $a_k(T) \to 0$ implies that T is compact. The converse implication generally does not hold.

$$e_k\Big(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\Big) \sim ?$$

$$a_k\Big(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\Big) \sim ?$$

- Let $d \geq 2$, $s \in \mathbb{R}$, $1 < p \leq \infty$ and $0 < q \leq \infty$. Then the space $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ is isomorphic to $R_G B^s_{p,q}(\mathbb{R}^m, w_\gamma)$, where G is some finite group of reflections acting on $\mathbb{R}^m$ and

$$w_\gamma(r_1, \ldots, r_m) = \prod_{i=1}^{m} |r_i|^{\gamma_i - 1}. \tag{3}$$

$$e_k\Big(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\Big) \sim ?$$

$$a_k\Big(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\Big) \sim ?$$

- Let $d \geq 2$, $s \in \mathbb{R}$, $1 < p \leq \infty$ and $0 < q \leq \infty$. Then the space $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ is isomorphic to $R_G B^s_{p,q}(\mathbb{R}^m, w_\gamma)$, where G is some finite group of reflections acting on $\mathbb{R}^m$ and

$$w_\gamma(r_1, \dots, r_m) = \prod_{i=1}^{m} |r_i|^{\gamma_i - 1}. \tag{3}$$

- We can reduce investigation of an asymptotic behaviour of entropy and approximation numbers of the studied embeddings to estimations for the corresponding weighted spaces with the Muckenhoupt weight w_γ, cf. (3), i.e.

$$e_k(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)) \sim \sim e_k(id : R_G B^{s_1}_{p_1,q_1}(\mathbb{R}^m, w_\gamma) \to R_G B^{s_2}_{p_2,q_2}(\mathbb{R}^m, w_\gamma)). \quad (4)$$

- Furthermore using the wavelet characterization of Besov spaces with $\mathcal{A}_\infty$ weights we can use the technique of discretization i.e., we can reduce the problem to the corresponding problem for suitable sequence spaces.

- We can reduce investigation of an asymptotic behaviour of entropy and approximation numbers of the studied embeddings to estimations for the corresponding weighted spaces with the Muckenhoupt weight w_γ, cf. (3), i.e.

$$e_k(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)) \sim \qquad (4)$$
$$\sim e_k(id : R_G B^{s_1}_{p_1,q_1}(\mathbb{R}^m, w_\gamma) \to R_G B^{s_2}_{p_2,q_2}(\mathbb{R}^m, w_\gamma)).$$

- Furthermore using the wavelet characterization of Besov spaces with $\mathcal{A}_\infty$ weights we can use the technique of discretization i.e., we can reduce the problem to the corresponding problem for suitable sequence spaces.

The main results

Theorem

Let $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$, $d = \gamma_1 + \ldots + \gamma_m$, $m \in \mathbb{N}$. *Let* $1 < p_1 < p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ *and* $s_1, s_2 \in \mathbb{R}$. *If* $\min_i \gamma_i \geq 2$ *and* $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$ *then*

$$e_k\left(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\right) \sim k^{-\frac{d}{m}(\frac{1}{p_1} - \frac{1}{p_2})}$$

and

$$a_k\Big(id : R_\gamma B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \to R_\gamma B^{s_2}_{p_2,q_2}(\mathbb{R}^d)\Big) \sim k^{-\alpha}$$

with

$$\alpha = \frac{d-m}{m}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \quad \text{if} \quad 1 < p_1 < p_2 \le 2 \quad \text{or} \quad 2 \le p_1 < p_2 \le \infty,$$

$$\alpha = \frac{d-m}{m}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) + \frac{1}{2} - \frac{1}{\min(p_1', p_2)} \quad \text{if} \quad 1 < p_1 < 2 < p_2 \le \infty$$
$$\text{and} \quad \frac{d-m}{m}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) > \frac{1}{\min(p_1', p_2)},$$

$$\alpha = \frac{d-m}{m}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \cdot \frac{\min(p_1', p_2)}{2} \quad \text{if} \quad 1 < p_1 < 2 < p_2 \le \infty$$
$$\text{and} \quad \frac{d-m}{m}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) < \frac{1}{\min(p_1', p_2)}.$$

Remark

If $m = 1$ then the spaces $R_\gamma B^s_{p,q}(\mathbb{R}^d)$ consists of radial distributions and the above estimates coincides with the estimates for radial functions. If $m < d$ and $\gamma_i \neq \gamma_j$ for some i and j then

$$\min_i \gamma_i < \frac{d}{m} < \max_i \gamma_i .$$

Thus the entropy numbers for spaces of block-radial functions goes asymptotically to zero quicker than the entropy numbers for radial function belonging to the corresponding spaces defined on the smallest block $\mathbb{R}^{\gamma_i}$. On the other hand the entropy numbers for spaces of block-radial functions goes asymptotically to zero slower than the entropy numbers for radial function belonging to the corresponding spaces defined on the biggest block $\mathbb{R}^{\gamma_i}$. Similarly for the approximation numbers.

Applicationes

We apply the asymptotic behaviour of the entropy numbers to a negative spectrum of Schrödinger type operators with block-radial potentials.

$$H_{s,\theta,\beta} \;=\; (\theta\,\mathrm{Id} - \Delta)^{s/2} - \beta V \tag{5}$$

where $s > 0$, $\beta > 0$ and $V \geq 0$ is an $SO(\gamma)$-invariant potential.
We are interested in a number of negative eigenvalues of $H_{s,\beta,\theta}$ with $SO(\gamma)$-invariant eigenfunctions. We put

$$N_{\gamma,\beta} = \#\Big\{\lambda \,:\, \lambda \leq 0\,, \quad H_{s,\theta,\beta} f = \lambda f\,, \quad f \in R_\gamma L_2(\mathbb{R}^d)\,,\ f \neq 0\,\Big\}.$$

$$N_{\gamma,\beta} \;\leq\; c\,\beta^{\frac{rm}{d}}.$$

Applicationes

We apply the asymptotic behaviour of the entropy numbers to a negative spectrum of Schrödinger type operators with block-radial potentials.

$$H_{s,\theta,\beta} \ = \ (\theta \,\mathrm{Id} - \Delta)^{s/2} - \beta V \tag{5}$$

where $s > 0$, $\beta > 0$ and $V \geq 0$ is an $SO(\gamma)$-invariant potential.
We are interested in a number of negative eigenvalues of $H_{s,\beta,\theta}$ with $SO(\gamma)$-invariant eigenfunctions. We put

$$N_{\gamma,\beta} = \#\Big\{\lambda \,:\, \lambda \leq 0\,, \quad H_{s,\theta,\beta} f = \lambda f\,, \quad f \in R_\gamma L_2(\mathbb{R}^d)\,,\ f \neq 0\Big\}.$$

$$N_{\gamma,\beta} \ \leq \ c\,\beta^{\frac{rm}{d}}.$$

Applicationes

We apply the asymptotic behaviour of the entropy numbers to a negative spectrum of Schrödinger type operators with block-radial potentials.

$$H_{s,\theta,\beta} \;=\; (\theta\,\mathrm{Id} - \Delta)^{s/2} - \beta V \tag{5}$$

where $s > 0$, $\beta > 0$ and $V \geq 0$ is an $SO(\gamma)$-invariant potential.
We are interested in a number of negative eigenvalues of $H_{s,\beta,\theta}$ with $SO(\gamma)$-invariant eigenfunctions. We put

$$N_{\gamma,\beta} = \#\Big\{\lambda \,:\, \lambda \leq 0\,, \quad H_{s,\theta,\beta} f = \lambda f\,, \quad f \in R_\gamma L_2(\mathbb{R}^d)\,,\; f \neq 0\,\Big\}.$$

$$N_{\gamma,\beta} \;\leq\; c\,\beta^{\frac{rm}{d}}\,.$$

References

A. Dota, L.Skrzypczak, *Some properties of block-radial functions and Schrödinger type operators with block-radial potentials* (prepared for printing)

Thank you for your attention.