Embeddings and characterizations of function spaces with logarithmic smoothness

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joint work with F. Cobos (Madrid), S. Tikhonov (Barcelona), H. Triebel (Jena)

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Introduction

Function spaces of generalized smoothness are useful to get the complete solution of some natural questions such as compactness of limiting embeddings, fractal analysis and a related spectral theory, probability theory and the theory of stochastic processes.

- ▻ M.L. Gol'dman, Trudy Mat. Inst. Steklov **156** (1980), 47 – 81.
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Assume $f \in W_p^{1+d/p}(\mathbb{R}^d), 1 < p < \infty$. Then, for all $x, y \in \mathbb{R}^d$ with $|x - y| < 1/2$,

$$|f(x) - f(y)| \leqslant c|x - y|\,|\log|x - y||^{1/p'} \|f\|_{W_p^{1+d/p}(\mathbb{R}^d)}.$$

▻ H. Brézis, S. Wainger, Comm. Part. Diff. Equ. **5** (1980), 773 – 789.

Fourier-analytical approach

Let $1 < p < \infty, 0 < q \leqslant \infty$ and $-\infty < s, b < \infty$. The Besov space $B^{s,b}_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{B^{s,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} (2^{js}(1+j)^b \|(\varphi_j \widehat{f})^\vee\|_{L_p(\mathbb{R}^d)})^q\right)^{1/q} < \infty.$$

Here, $\{\varphi_j\}_{j=0}^{\infty}$ denotes a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty} \varphi_j(x) = 1, x \in \mathbb{R}^d$.

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The Sobolev space $W^{s,b}_p(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W^{s,b}_p(\mathbb{R}^d)} = \|((1+|x|^2)^{s/2}(1+\log(1+|x|^2))^b\widehat{f})^\vee\|_{L_p(\mathbb{R}^d)} < \infty.$$

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If $b = 0$ we get the fractional Sobolev spaces $W^s_p(\mathbb{R}^d)$. In addition, if $s = k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$$\|f\|_{W^k_p(\mathbb{R}^d)} \asymp \sum_{|\alpha|\leqslant k}\|D^\alpha f\|_{L_p(\mathbb{R}^d)}.$$

Differences

For $k \in \mathbb{N}$ and $h, x \in \mathbb{R}^d$, we put

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \text{ and } \Delta_h^{k+1} f = \Delta_h^1(\Delta_h^k f).$$

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The k-th order modulus of smoothness of $f \in L_p(\mathbb{R}^d)$ is given by

$$\omega_k(f,t)_p = \sup_{|h| \leqslant t} \|\Delta_h^k f\|_{L_p(\mathbb{R}^d)}, \quad t > 0.$$

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Assume that $0 \leqslant s < k \in \mathbb{N}$. The Besov space $\mathbb{B}_{p,q}^{s,b}(\mathbb{R}^d)$ consists of all $f \in L_p(\mathbb{R}^d)$ having finite quasi-norm

$$\|f\|_{\mathbb{B}_{p,q}^{s,b}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (t^{-s}(1 - \log t)^b \omega_k(f,t)_p)^q \frac{dt}{t} \right)^{1/q}.$$

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If $b = 0$ we recover the classical spaces $\mathbb{B}^s_{p,q}(\mathbb{R}^d)$. If $s = 0$ the case of interest is when $b \geqslant -1/q$. Otherwise, $\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.

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$$\begin{aligned}\|f\|_{\mathbb{B}_{p,q}^{s,b}(\mathbb{R}^d)} &= \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (t^{-s}(1-\log t)^b \omega_k(f,t)_p)^q \frac{dt}{t}\right)^{1/q} \\ &\asymp \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h| \leqslant 1} (|h|^{-s}(1-\log|h|)^b \|\Delta_h^k f\|_{L_p(\mathbb{R}^d)})^q \frac{dh}{|h|^{d+1}}\right)^{1/q}.\end{aligned}$$

Embeddings between Besov, Triebel-Lizorkin and Sobolev spaces

- *F*- and *W*-spaces

Let us recall that

$$\|f\|_{F^0_{p,2}(\mathbb{R}^d)} = \left\| \left(\sum_{j=0}^{\infty} |(\varphi_j \widehat{f})^\vee(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \asymp \|f\|_{L_p(\mathbb{R}^d)} \text{ (Littlewood-Paley),}$$

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▻ W. Sickel, H. Triebel, Z. Anal. Anwendungen **14** (1995), 105 –140.

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More contributions ∼80's

▻ R.A. DeVore, S.D. Riemenschneider, R.C. Sharpley, J. Funct. Anal. **33** (1979), 58–94.

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◇ Approximation theory

▻ B.S. Kashin, V.N. Temlyakov, J. Math. Sci. **155** (2008), 57 – 80.

▻ K. Runovski, H.-J. Schmeisser.

▻ F. Cobos, O. Domínguez, T. Kühn, Constr. Approx. (to appear).

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REMARK

Let $1 < p < \infty$. For $t > 0$,

$$\Big\| f - \frac{1}{|B_t(x)|} \int_{|x-y| \leqslant t} f(y) dy \Big\|_{L_p(\mathbb{R}^d)} \asymp \inf_{\Delta g \in L_p(\mathbb{R}^d)} \|f - g\|_{L_p(\mathbb{R}^d)} + t^2 \|\Delta g\|_{L_p(\mathbb{R}^d)}$$

Ditzian, Dai, Runovskii, ...

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$$\left\| f - \frac{1}{|B_t(x)|} \int_{|x-y| \leqslant t} f(y)dy \right\|_{L_p(\mathbb{R}^d)} \asymp \omega_2(f,t)_p$$

Let $p = 1$. Let $A \subset \mathbb{R}^d$ be a measurable subset. If $f = \chi_A$ then

$$\|f\|_{\mathbb{B}^0_{1,1}(\mathbb{R}^d)} \sim \|f\|_{L_1(\mathbb{R}^d)} + \int_0^1 \left\| f - \frac{1}{|B_t(x)|} \int_{|x-y| \leqslant t} f(y)dy \right\|_{L_1(\mathbb{R}^d)} \frac{dt}{t}$$

The theory of Besov spaces $\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d)$ is attracting a great interest in recent times.

◇ Trace and extension theorems

▻ L. Malý, N. Shanmugalingam, M. Snipes, Ann. Scuola Norm. Sup. Pisa (to appear).

◇ Problems on mixing flows

▻ S. Bianchini, Banach Center Publ. **74** (2006), 13–31.

▻ M. Hadžić, A. Seeger, C.K. Smart, B. Street, Ann. Inst. H. Poincaré Anal. Non Linéaire (to appear).

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The corresponding results for periodic spaces and averages on spheres also hold.

$$\|f\|_{F^0_{p,2}(\mathbb{R}^d)} = \left\| \left(\sum_{\nu=0}^{\infty} |(\varphi_\nu \widehat{f})^\vee(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \asymp \|f\|_{L_p(\mathbb{R}^d)} \text{ (Littlewood-Paley).}$$

Assume that $s = 0$. Then,

$$\|f\|_{\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d)} \not\asymp \|f\|_{B^{0,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} (1+j)^{bq} \|(\varphi_j \widehat{f})^\vee\|^q_{L_p(\mathbb{R}^d)} \right)^{1/q}.$$

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▻ F. Cobos, O. Domínguez, H. Triebel, J. Funct. Anal. **270** (2016), 4386–4425.

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Characterizations of $\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d)$ in terms of wavelets, heat kernels, harmonic extensions, ...

Assume that $b > -1/q$. Then we have

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

In particular, it holds that

$$\mathbb{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d) \text{ for } b > -1/2.$$

▻ F. Cobos, O. Domínguez, J. Math. Anal. Appl. **425** (2015) 71–84.

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Are these embeddings sharp?

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Are these embeddings sharp?

Can these embeddings be improved if we restrict ourselves to some important classes of functions?

- *F*- and *B*-spaces

Let $1 < p < \infty, 0 < q, r \leqslant \infty$ and $-\infty < s, b < \infty$. The following embeddings are well known

$$F^{s,b}_{p,r}(\mathbb{R}^d) \hookrightarrow B^{s,b}_{p,q}(\mathbb{R}^d) \text{ for } q \geqslant \max\{p, r\}.$$

▷ H. Triebel, Birkhäuser, Basel, 1983.

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In particular, if $s > 0$ then

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Moreover, this embedding is sharp.

Characterizations and embeddings theorems for general monotone functions

There are many problems in analysis where monotonicity of a sequence or a function plays a crucial role. For instance, convergence results for Fourier series, weighted integrability of the Fourier transform and the Paley-Wiener theorem on integrability of the function conjugate of an odd one.

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A complex-valued function $\varphi(z), z > 0$, is said to be general monotone if it is locally of bounded variation and there exists a constant $C > 1$ such that

$$\int_z^{2z} |d\varphi(u)| \leqslant C|\varphi(z)|, \quad z > 0.$$

We denote by GM the class of all general monotone functions.

▻ S. Tikhonov, J. Math. Anal. Appl. **326** (2007), 721 – 735.

▻ E. Liflyand, S. Tikhonov, Math. Nachr. **284** (2011), 1083 – 1098.

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EXAMPLES

- Decreasing functions.
- Quasi-monotone functions: $t^{-\alpha}\varphi(t)$ is non-increasing for some $\alpha \geqslant 0$.
- Increasing functions satisfying that $\varphi(2t) \lesssim \varphi(t), t > 0$.

The Fourier transform of a radial function $f(x) = f_0(|x|), x \in \mathbb{R}^d$, is also radial $\widehat{f}(\xi) = F_0(|\xi|)$ where F_0 is the Hankel-Fourier transform of f_0.

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We denote by $\widehat{GM}^d$ the collection of all radial functions f such that F_0 is non-negative, $F_0 \in GM$ and satisfies the condition

$$\int_0^1 t^{d-1} F_0(t)dt + \int_1^\infty t^{(d-1)/2} |dF_0(t)| < \infty.$$

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Hardy-Littlewood theorem for $\widehat{GM}^d$ class: Assume that $f \in \widehat{GM}^d$. Then,

$$\|f\|_{L_p(\mathbb{R}^d)} \asymp \left(\int_0^\infty F_0^p(t) t^{dp-d-1} dt \right)^{1/p}, \quad \frac{2d}{d+1} < p < \infty.$$

▻ D. Gorbachev, E. Liflyand, S. Tikhonov, J. Anal. Math. **114** (2011), 99–120.

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PROBLEM: Characterizations of smoothness spaces in terms of the growth properties of the Fourier transform.

THEOREM (D., Tikhonov).- Let $\frac{2d}{d+1} < p < \infty$ and $0 < q \leqslant \infty$. Let $f \in \widehat{GM}^d$.

(i) If $s > 0$ and $-\infty < b < \infty$, then

$$\|f\|_{\mathbb{B}^{s,b}_{p,q}(\mathbb{R}^d)} \asymp \left(\int_0^1 t^{dp-d-1} F_0^p(t)dt\right)^{1/p} + \left(\int_1^\infty t^{sq+dq-dq/p-1}(1+\log t)^{bq} F_0^q(t)dt\right)^{1/q}.$$

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We need to introduce truncated Hardy-Littlewood constructions.

$$\|f\|_{L_p(\mathbb{R}^d)} \asymp \left(\int_0^\infty u^{dp-d-1} F_0(u)^p du\right)^{1/p}.$$

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We recall that if $b > -1/q$ then

$$B_{p,q}^{0,b+1/\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{0,b+1/\max\{2,p,q\}}(\mathbb{R}^d).$$

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We recall that

$$W_p^{s,b}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{s,b}(\mathbb{R}^d) \text{ and } W_p^{0,b+1/q}(\mathbb{R}^d) \hookrightarrow \mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d) \text{ for } q \geqslant \max\{p, 2\}.$$

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Moreover, this embedding is sharp: given any $\varepsilon > 0$ there exists $f \in \widehat{GM}^d$ such that $f \in W_p^{0,b+1/q-\varepsilon}(\mathbb{R}^d)$ but $f \notin \mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d)$.

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(ii) If $b > -1/q$ then

$$\widehat{GM}^d \cap W_p^{0,b+1/q}(\mathbb{R}^d) \hookrightarrow \widehat{GM}^d \cap \mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d) \text{ for } q \geqslant p.$$

Moreover, this embedding is sharp: given any $\varepsilon > 0$ there exists $f \in \widehat{GM}^d$ such that $f \in W_p^{0,b+1/q-\varepsilon}(\mathbb{R}^d)$ but $f \notin \mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d)$.

COROLLARY (D., Tikhonov).- Let $\frac{2d}{d+1} < p < \infty$.

(i) If $-\infty < s, b < \infty$ then

$$\widehat{GM}^d \cap W_p^{s,b}(\mathbb{R}^d) = \widehat{GM}^d \cap B_{p,p}^{s,b}(\mathbb{R}^d).$$

(ii) If $b > -1/p$ then

$$\widehat{GM}^d \cap W_p^{0,b+1/p}(\mathbb{R}^d) = \widehat{GM}^d \cap \mathbb{B}_{p,p}^{0,b}(\mathbb{R}^d).$$

The Franke-Jawerth embedding for Besov and Sobolev spaces states that if $1 < p_0 < p < p_1 < \infty, -\infty < s_1 < s < s_0 < \infty$ with

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}$$

and $-\infty < b < \infty$, then

$$B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \hookrightarrow W^{s,b}_p(\mathbb{R}^d) \hookrightarrow B^{s_1,b}_{p_1,p}(\mathbb{R}^d).$$

▻ B. Jawerth, Math. Scand. **40** (1977), 94 – 104.

▻ J. Franke, Math. Nachr. **125** (1986), 29 – 68.

$$B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \hookrightarrow W^{s,b}_p(\mathbb{R}^d) \hookrightarrow B^{s_1,b}_{p_1,p}(\mathbb{R}^d)$$

For $f \in \widehat{GM}^d$ and $\tau > 0$, we put

$$J_f(\tau) = \left(\int_0^1 t^{d\tau - d - 1} F_0^\tau(t)\, dt \right)^{1/\tau}.$$

THEOREM (D., Tikhonov).- Let
$\frac{2d}{d+1} < p_0 < p < p_1 < \infty, -\infty < s_1 < s < s_0 < \infty$ with

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}$$

and $-\infty < b < \infty$. Let $f \in \widehat{GM}^d$.

(i) If $J_f(p_0) < \infty$ then

$$f \in B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \iff f \in W^{s,b}_p(\mathbb{R}^d).$$

(ii) If $J_f(p) < \infty$ then

$$f \in W^{s,b}_p(\mathbb{R}^d) \iff f \in B^{s_1,b}_{p_1,p}(\mathbb{R}^d).$$

$$B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \hookrightarrow W^{s,b}_p(\mathbb{R}^d) \hookrightarrow B^{s_1,b}_{p_1,p}(\mathbb{R}^d)$$

For $f \in \widehat{GM}^d$ and $\tau > 0$, we put

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$\frac{2d}{d+1} < p_0 < p < p_1 < \infty, -\infty < s_1 < s < s_0 < \infty$ with

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and $-\infty < b < \infty$. Let $f \in \widehat{GM}^d$.

(i) If $J_f(p_0) < \infty$ then

$$f \in B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \iff f \in W^{s,b}_p(\mathbb{R}^d).$$

(ii) If $J_f(p) < \infty$ then

$$f \in W^{s,b}_p(\mathbb{R}^d) \iff f \in B^{s_1,b}_{p_1,p}(\mathbb{R}^d).$$

The conditions $J_f(p_0) < \infty$ and $J_f(p) < \infty$ given in (i) and (ii), respectively, are necessary.

$$B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \hookrightarrow W^{s,b}_p(\mathbb{R}^d) \hookrightarrow B^{s_1,b}_{p_1,p}(\mathbb{R}^d)$$

For $f \in \widehat{GM}^d$ and $\tau > 0$, we put

$$J_f(\tau) = \left(\int_0^1 t^{d\tau-d-1} F_0^\tau(t)\,dt\right)^{1/\tau}.$$

THEOREM (D., Tikhonov).- Let
$\frac{2d}{d+1} < p_0 < p < p_1 < \infty, -\infty < s_1 < s < s_0 < \infty$ with

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}$$

and $-\infty < b < \infty$. Let $f \in \widehat{GM}^d$.

(i) If $J_f(p_0) < \infty$ then

$$f \in B^{s_0,b}_{p_0,p}(\mathbb{R}^d) \iff f \in W^{s,b}_p(\mathbb{R}^d).$$

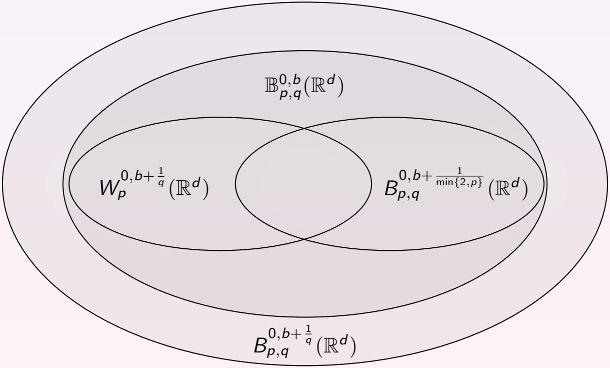
(ii) If $J_f(p) < \infty$ then

$$f \in W^{s,b}_p(\mathbb{R}^d) \iff f \in B^{s_1,b}_{p_1,p}(\mathbb{R}^d).$$

The conditions $J_f(p_0) < \infty$ and $J_f(p) < \infty$ given in (i) and (ii), respectively, are necessary.
We also obtain the corresponding result for Sobolev embeddings $B^{s_0,b}_{p_0,q}(\mathbb{R}^d) \hookrightarrow B^{s_1,b}_{p_1,q}(\mathbb{R}^d)$.

Relationships between function spaces with smoothness near zero

$q \geqslant \max\{p, 2\}$



Sharpness assertions

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d), b > -1/q.$$

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THEOREM (D., Tikhonov).- Let $\frac{2d}{d+1} < p < \infty, 1 \leqslant q \leqslant \infty$ and $b > -1/q$. Then, for any $\varepsilon > 0$ we have

$$B^{0,b+1/\min\{2,p,q\}-\varepsilon}_{p,q}(\mathbb{R}^d) \not\hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d)$$

and

$$\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/\max\{2,p,q\}+\varepsilon}_{p,q}(\mathbb{R}^d).$$

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$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d), b > -1/q.$$

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and

$$\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/\max\{2,p,q\}+\varepsilon}_{p,q}(\mathbb{R}^d).$$

THEOREM.- Let $\frac{2d}{d+1} < p < \infty, 0 < q \leqslant \infty, b > -1/q$ and $-\infty < \xi < \infty$. Then,

$$\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) = B^{0,\xi}_{p,q}(\mathbb{R}^d) \Longleftarrow p = q = 2 \quad \text{and} \quad \xi = b + \frac{1}{2}.$$

Sharpness assertions

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d), b > -1/q.$$

THEOREM (D., Tikhonov).- Let $\frac{2d}{d+1} < p < \infty, 1 \leqslant q \leqslant \infty$ and $b > -1/q$. Then, for any $\varepsilon > 0$ we have

$$B^{0,b+1/\min\{2,p,q\}-\varepsilon}_{p,q}(\mathbb{R}^d) \not\hookrightarrow \mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d)$$

and

$$\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/\max\{2,p,q\}+\varepsilon}_{p,q}(\mathbb{R}^d).$$

THEOREM (D., Tikhonov) Let $\frac{2d}{d+1} < p < \infty, 0 < q \leqslant \infty, b > -1/q$ and $-\infty < \xi < \infty$. Then,

$$\mathbb{B}^{0,b}_{p,q}(\mathbb{R}^d) = B^{0,\xi}_{p,q}(\mathbb{R}^d) \iff p = q = 2 \quad \text{and} \quad \xi = b + \frac{1}{2}.$$