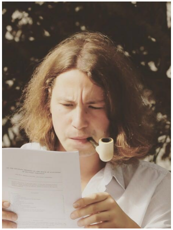
Universität Bielefeld

Boundary Values of Functions with Bounded Deviatoric Variation

Lars Diening

Dominic Breit



Franz Gmeineder

Problem: Functionals with Linear Growth

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Minimize $\mathfrak{F}(u) = \int_\Omega f(x, \nabla u(x))\, dx$ subject to Dirichlet data u_0.

linear growth: $c_1|\nabla u| \leq f(\cdot, \nabla u) \leq c_2|\nabla u| + c_3$

continuity: $|f(x, Q) - f(y, Q)| \leq \omega(|x - y|)(1 + |Q|)$.

quasi-convexity: linear functions are local minimizers.

Singularities (line discontinuities) will develop!

Questions

Functions spaces: Compactness

Boundary values: Existence of traces

Now, replace ∇u by a first order differential operator $\mathbb{A}$.

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Differential Operator – Main Examples

Let $u : \mathbb{R}^n \to \mathbb{R}^N$ and $N(\mathbb{A}) = \{u : \mathbb{A}u = 0 \text{ as distribution}\}$.

Gradient

$\mathbb{A}u = \nabla u, \quad N(\nabla) = \{x \mapsto b\}$.

Symmetric Gradient ($n = N \geq 2$)

$\mathbb{A} = \mathcal{E}u = \frac{1}{2}((\nabla u) + (\nabla u)^T)$,

$N(\mathcal{E}) = \{x \mapsto Ax + b : A + A^T = 0\}$.

Trace Free Gradient ($n = N \geq 2$)

$\mathbb{A} = \mathcal{E}^D u = \mathcal{E}u - \frac{1}{n}\mathrm{Id}\,\mathrm{div}\,u$.

$n = 2$: $\quad N(\mathcal{E}^D) = \texttt{holomorphic functions}$.

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The Function Spaces

Sobolev Space

$$W^{\mathbb{A},1}(\Omega) := \{u \in L^1_{\mathrm{loc}} u \,:\, \mathbb{A}u \in L^1(\Omega)\}.$$
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Problem: No weak-$*$ compactness on L^1!

Solution: $L^1(\Omega) \subset \mathcal{M}(\Omega)$ (space of Radon measures)

and $\mathcal{M}(\Omega) = (C^0(\Omega))^*$.

$\Rightarrow$ weak-$*$ compactness.

Space of Bounded Variation

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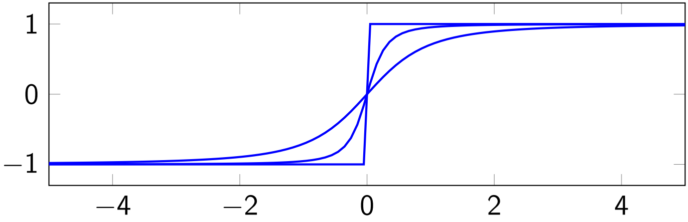

Higher dimensions:

Take indicator function χ_B of ball B, then $\chi_B \in \mathrm{BV}^{\nabla}(\Omega)$ and

$$\nabla \chi_B = -\nu \mathcal{H}^{n-1} \llcorner_{\partial B}.$$

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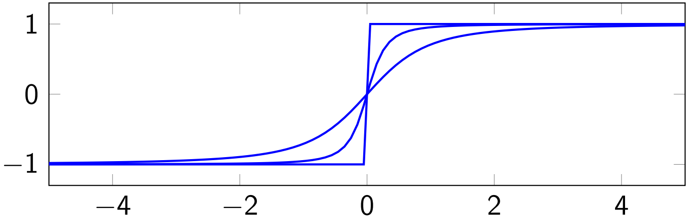

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If $1 < p < \infty$, then $W^{1,p}(\Omega) \approx W^{\mathcal{E},p}(\Omega)$.

Ornstein's Non-Inequality

$$W^{1,1}(\Omega) \subsetneq W^{\mathcal{E},1}(\Omega).$$

Additional properties:

Poincaré: $\|u - \Pi u\|_{L^1(B_r)} \lesssim r\,\|\mathcal{E}u\|_{L^1(B_r)}$.

Embedding: $\|u - \Pi u\|_{L^{\frac{n}{n-1}}(B_r)} \lesssim \|\mathcal{E}u\|_{L^1(B_r)}$.

Same for BV-version.

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Classical Traces

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There exists a trace operator $\mathrm{tr} : W^{1,1}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$.

By density it suffices to show estimates for smooth u.

Estimate is based on fundamental theorem of calculus:

$$u(y) - u(x) = \int_0^1 (\nabla u)\big(x + t(y-x)\big) \cdot (y-x)\, dt.$$

Choose $y \in \partial\Omega$ and $x \to \infty$.

Trace of Functions with Bounded Variation

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Traces for Functions of Bounded Deformation

BV-Trace [Strang-Teman '81, Babadjian '13]

There exists a trace operator

$$\begin{aligned} \mathrm{tr}\ : W^{\mathcal{E},1}(\Omega) &\to L^1(\partial\Omega, \mathcal{H}^{n-1}), \\ \mathrm{BV}^{\mathcal{E}}(\Omega) &\to L^1(\partial\Omega, \mathcal{H}^{n-1}). \end{aligned}$$

Idea: For every direction $a \in \mathbb{R}^n$

$$\begin{aligned} \big(u(x+a) - u(x)\big) \cdot a &= \int_0^1 (\nabla u)\big(x + ta\big) : \big(a \otimes a\big)\, dt \\ &= \int_0^1 (\mathcal{E} u)\big(x + a\big) : \big(a \otimes a\big)\, dt. \end{aligned}$$

Enough to recover u_n with $a = e_n$ and then the other u_j.

A Counterexample

Consider $\mathbb{A} = \mathcal{E}^D$ for $n = N = 2$. Then with $\mathbb{R}^2 = \mathbb{C}$

$$\mathcal{N}(\mathcal{E}^D) = \{u \,:\, u \text{ holomorphic}\}.$$

$$\mathcal{E}^D(u) = \frac{1}{2}\begin{pmatrix} \partial_1 u_1 - \partial_2 u_2 & \partial_1 u_2 + \partial_2 u_1 \\ \partial_1 u_2 + \partial_2 u_1 & \partial_1 u_1 - \partial_2 u_2 \end{pmatrix}$$

Let $u(z) := \frac{1}{z}$, then $\mathcal{E}^D u = 0 \in \mathcal{D}'(\mathbb{R}^2 \setminus \{0\})$.

Hence, $u \in W^{1,\mathcal{E}^D}(B_1(1))$, but $u \notin L^1(\partial B_1(1))$.

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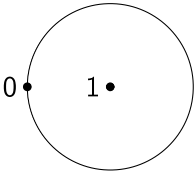

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Ellipticity

For $\mathbb{A} = \sum_{\alpha=1}^{n} \mathbb{A}_\alpha \partial_\alpha$ with $\mathbb{A}_\alpha \in L(\mathbb{R}^N; \mathbb{R}^K)$

define the *symbol mapping* $\mathbb{A}[\xi] : \mathbb{R}^N \to \mathbb{R}^K$ by

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^{n} \xi_\alpha \mathbb{A}_\alpha v.$$

Example: $v \otimes_\nabla \xi = v \otimes \xi$.

Definition

$\mathbb{A}$ is $\mathbb{R}$-*elliptic* if $\mathbb{A}[\xi] : \mathbb{R}^N \to \mathbb{R}^K$ is injective for all $\xi \neq 0$.

$\mathbb{A}$ is $\mathbb{C}$-*elliptic* if $\mathbb{A}[\xi] : \mathbb{C}^N \to \mathbb{C}^K$ is injective for all $\xi \neq 0$.

∇, $\mathcal{E}$ and $\mathcal{E}^D$ are $\mathbb{R}$-elliptic.

They are also $\mathbb{C}$-elliptic with the exception: $\mathcal{E}^D$ for $n = N = 2$.

A Simple Characterization

Recall $N(\mathbb{A}) = \{u : \mathbb{A}u = 0 \text{ as distribution}\}$.

Characterization

The following are equivalent

- $\mathbb{A}$ is $\mathbb{C}$-elliptic.
- $N(\mathbb{A})$ is finite dimensional.
- $N(\mathbb{A})$ is a finite dimensional set of polynomials.

Observation:

$\mathbb{A}$	L^1-trace	$\mathbb{C}$-elliptic
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The Work of Agnieszka Kałamajska '94

Kałamaskja considered differential operators also of higher order.

$\mathbb{A}$ is $\mathbb{C}$-elliptic. $\Leftrightarrow$ $\mathbb{A}$ is of type (C) of Kałamasjka.

Representation formula on ball B

$$|u(x) - (\Pi u)(x)| \lesssim \int_B \frac{|\mathbb{A}u)(y)|}{|x-y|^{n-1}}\,dy,$$

where Π is averaged Taylor polynomial of suitable high order

or its projection to $N(\mathbb{A})$.

Idea: $\nabla^{l-1}\mathbb{A} \eqsim \nabla^l$ for large l.

Immediate consequence:

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Trace Theorem [Breit, Diening, Gmeineder '17]

Let Ω be bounded with $\partial\Omega$ Lipschitz and let $\mathbb{A}$ be $\mathbb{C}$-elliptic.

Then there exists a trace operator $\mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$.

It is continuous with respect to *strict convergence*, i.e. for

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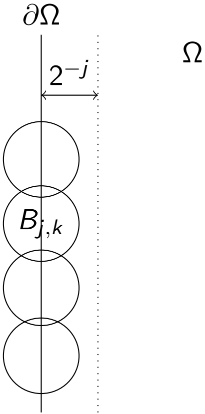

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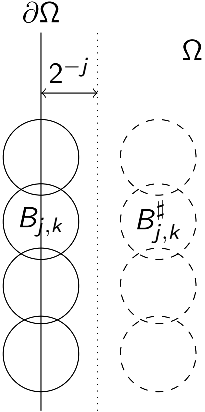

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Project u on $B^{\sharp}_{j,k}$ to $\Pi_{j,k}u \in N(\mathbb{A})$ and

replace u on $B_{j,k}$ by $\Pi_{j,k}u$ i.e.

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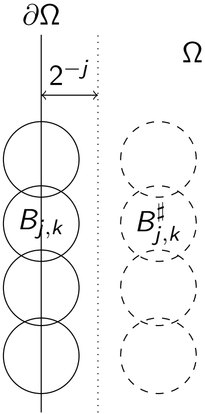

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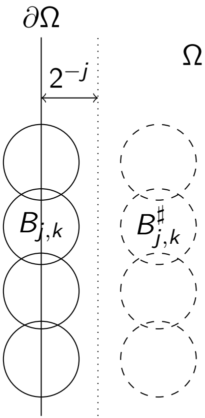
Then $T_j u \to u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$

and $\mathrm{tr}(T_j u) \to: \mathrm{tr}(u)$ in $L^1(\partial\Omega)$.

Based on inverse estimates for polyomials!



Consequences

Gauß-Green Fomula

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^K)$ we have

$$\int_\Omega \mathbb{A}u \cdot \varphi \, dx = -\int_\Omega u \cdot \mathbb{A}^* \varphi \, dx + \int_{\partial\Omega} (\mathrm{tr}(u) \otimes_{\mathbb{A}} \nu) \cdot \varphi \, d\mathcal{H}^{n-1}.$$

This allows gluing of $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$.

Gluing

Glue $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$ together by

$$w := \chi_\Omega u + \chi_{U \setminus \Omega} v.$$

Then $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and

$$\mathbb{A}w = \mathbb{A}u \llcorner_\Omega + \mathbb{A}v \llcorner_{U \setminus \Omega} + (\mathrm{tr}^+(v) - \mathrm{tr}^-(u)) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}.$$

Consequences

Gauß-Green Fomula

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^K)$ we have

$$\int_\Omega \mathbb{A}u \cdot \varphi \, dx = -\int_\Omega u \cdot \mathbb{A}^* \varphi \, dx + \int_{\partial\Omega} (\mathrm{tr}(u) \otimes_{\mathbb{A}} \nu) \cdot \varphi \, d\mathcal{H}^{n-1}.$$

This allows gluing of $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$.

Gluing

Glue $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$ together by

$$w := \chi_\Omega u + \chi_{U \setminus \Omega} v.$$

Then $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and

$$\mathbb{A}w = \mathbb{A}u \llcorner_\Omega + \mathbb{A}v \llcorner_{U \setminus \Omega} + (\mathrm{tr}^+(v) - \mathrm{tr}^-(u)) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}.$$

Back to Variational Integrals

Problem

Minimize $\mathfrak{F}(u) = \int_\Omega f\big(x, (\mathbb{A}u)(x)\big)\, dx$ subject to Dirichlet data u_0

linear growth: $c_1|\mathbb{A}u| \leq f(\cdot, \mathbb{A}u) \leq c_2|\mathbb{A}u| + c_3$

$+$ continuity $+$ quasi-convexity.

- Extend (relax) $\mathfrak{F}$ from $W^{1,\mathbb{A}}(\Omega)$ to $\mathrm{BV}^{\mathbb{A}}(\Omega)$.
  (Lower semi continuous envelope.)
- $f(x, \mathtt{measure})$ not defined!
  Need recession function f_∞ taking care of jumps!
  Recall $u_\varepsilon(x) = \frac{x}{\sqrt{\varepsilon^2+x^2}} \overset{*}{\rightharpoonup} \mathrm{sgn}(x) := u(x)$. Then $\nabla u = 2\delta_0$.
- Lavrientiev gap: $\inf \mathfrak{F}(W^{1,\mathbb{A}}(\Omega)) = \min \mathfrak{F}(\mathrm{BV}^{\mathbb{A}}(\Omega))$?

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Neccessary Tools: Van Schaftingen

For the variational tools we also need that $\mathbb{A}$ is *cancelling*, i.e.

$$\bigcap_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^N) = \{0\}.$$

Van Schaftingen '13: Necessary and sufficient for $\|u\|_{\frac{n}{n-1}} \lesssim \|\mathbb{A}u\|_1$.

Lemma [Gmeineder, Reita '17]

If $\mathbb{A}$ is $\mathbb{C}$-elliptic, then it is cancelling.

The reverse implication fails:

$$\mathbb{A}u := \begin{pmatrix} \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \partial_3 u_1 \\ \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \partial_3 u_2 \end{pmatrix}.$$

is cancelling but has infinite dimensional null space.

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Functionals with Linear Growth

Problem: $\mathfrak{F}(u) = \int_\Omega f\big(x, (\mathbb{A}u)(x)\big)\,dx$ subject to Dirichlet data u_0.

Recession function: $f_\infty(x, A) := \lim_{t\to\infty, x'\to x, A'\to A} \frac{f(x', tA')}{t}$.

Existence [Breit, Diening, Gmeineder '17]

The *extension* $\overline{\mathfrak{F}}_{\mathbf{u}_0} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ of $\mathfrak{F}$ is given by

$$\overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}] := \int_\Omega f\left(x, \frac{d\mathbb{A}\mathbf{u}}{d\mathcal{L}^n}\right) d\mathcal{L}^n + \int_\Omega f^\infty\left(x, \frac{d\mathbb{A}\mathbf{u}}{d|\mathbb{A}^s\mathbf{u}|}\right) d|\mathbb{A}^s\mathbf{u}|$$
$$+ \int_{\partial\Omega} f^\infty\Big(x, \nu_{\partial\Omega} \otimes_{\mathbb{A}} \mathrm{tr}(\mathbf{u} - \mathbf{u}_0)\Big)\, d\mathcal{H}^{n-1}.$$

Moreover, $\overline{\mathfrak{F}}_{\mathbf{u}_0}$ has a minimizer and $\min_{\mathrm{BV}^{\mathbb{A}}(\Omega)} \overline{\mathfrak{F}}_{u_0} = \inf_{u_0 + W_0^{1,\mathbb{A}}(\Omega)} \mathfrak{F}$.

Summary

Trace Theorem

There exists a trace operator $\mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$

if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic.

if and only if null space $N(\mathbb{A})$ is finite dimensional.

Application

The $\mathbb{A}$-variational problem with linear growth

and L^1-boundary data has a solution in $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

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Thank you for your attention.