# Maximal $L^p$-regularity for mixed-order systems in Sobolev spaces related to the Newton polygon

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    - The Stefan problem
    - The idea of maximal regularity
2. The Newton polygon approach
    - Definition of the Newton polygon
    - Maximal regularity results
3. Applications
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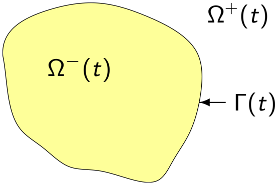
# The Stefan problem

Consider the Stefan problem with Gibbs-Thomson correction (free boundary problem)

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } \Omega^\pm(t)\ (t>0),\\ u &= \kappa \quad \text{on } \Gamma(t),\\ V &= [\partial_\nu u] \quad \text{on } \Gamma(t),\\ u|_{t=0} &= u_0 \quad \text{in } \Omega^\pm(0),\\ \Gamma(0) &= \Gamma_0. \end{aligned}$$




$\kappa$: sum of principal curvatures of $\Gamma(t)$,
$V$: normal velocity of $\Gamma(t)$,
$[\partial_\nu u]$: jump of normal derivatives.

# The idea of maximal regularity

Quasilinear parabolic problems can be written in the form

$$\begin{aligned} (\partial_t - A)u &= G(u) \quad (t > 0), \\ \gamma_t u &= u_0 \end{aligned}$$

Here, for some Banach spaces $\mathbb{E}$ and $\mathbb{F}$, the operator $\partial_t - A\colon \mathbb{E} \to \mathbb{F}$ is linear and bounded, $\gamma_t u := u|_{t=0}$, and $G \in C^1(\mathbb{E}, \mathbb{F})$ with $G'(0) = 0$.

# The idea of maximal regularity

$$\text{Consider} \quad (\partial_t - A)u = G(u)\,(t > 0),$$
$$\gamma_t u = u_0.$$

Maximal regularity: The operator $A$ is said to have maximal regularity if the linearized problem

$$L := \begin{pmatrix} \partial_t - A \\ \gamma_t \end{pmatrix} : \mathbb{E} \to \mathbb{F} \times \mathbb{E}_0$$

defines an isomorphism of Banach spaces. In this case the nonlinear equation is reduced to a fixed point problem

$$u = L^{-1} \begin{pmatrix} G(u) \\ u_0 \end{pmatrix}.$$

# Maximal $L^p$-regularity

Example: The Dirichlet Laplacian in a bounded smooth domain $G$ induces an isomorphism

$$u \mapsto \begin{pmatrix} (\partial_t - \Delta)u \\ u|_{\partial G} \\ u|_{t=0} \end{pmatrix} : \mathbb{E} \to \mathbb{F} \times \mathbb{G} \times \mathbb{E}_0$$

if we choose ($L^p$-setting)

$$\begin{aligned}
\mathbb{E} &:= W_p^1((0,T); L^p(G)) \cap L^p((0,T), W_p^2(G)), \\
\mathbb{F} &:= L^p((0,T), L^p(G)), \\
\mathbb{G} &:= B_{pp}^{1-1/(2p)}((0,T), L^p(\partial G)) \cap L^p((0,T), B_{pp}^{2-1/p}(\partial G)), \\
\mathbb{E}_0 &:= B_{pp}^{2-2/p}(G).
\end{aligned}$$

# The linearized Stefan problem

The above Stefan problem leads to the linearized model problem

$$\begin{aligned} (\partial_t - \Delta)u &= f \quad && \text{in } (0,T)\times\mathbb{R}^n_+,\\ u\big|_{\mathbb{R}^{n-1}} + \Delta'\,\sigma &= g && \text{in } (0,T)\times\mathbb{R}^{n-1},\\ -\partial_{x_n} u\big|_{\mathbb{R}^{n-1}} + \partial_t\,\sigma &= h && \text{in } (0,T)\times\mathbb{R}^{n-1}. \end{aligned}$$

- The spaces for $u$, $f$, $g$, and $h$ are standard. But what is the space for $\sigma$?

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- The spaces for $u$, $f$, $g$, and $h$ are standard. But what is the space for $\sigma$?

After Laplace transform in time and Fourier transform in tangential direction, one obtains a mixed-order pseudodifferential system on the boundary with symbol

$$L(\xi',\lambda) = \begin{pmatrix} 1 & -|\xi'|^2 \\ \sqrt{|\xi'|^2+\lambda} & \lambda \end{pmatrix}$$

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# The Newton polygon

The Lopatinskii matrix of the Stefan problem is given by

$$L(\xi', \lambda) = \begin{pmatrix} \frac{1}{\sqrt{|\xi'|^2+\lambda}} & -|\xi'|^2 \\ & \lambda \end{pmatrix}$$

with determinant

$$\det L(\xi', \lambda) = \lambda + |\xi'|^2 \sqrt{|\xi'|^2 + \lambda}.$$

Compare with the symbol of the heat equation: $A(\xi', \lambda) = \lambda + |\xi'|^2$.

# The Newton polygon

To understand symbols like $\lambda + |\xi'|^2\sqrt{|\xi'|^2+\lambda}$, one can use the Newton polygon:

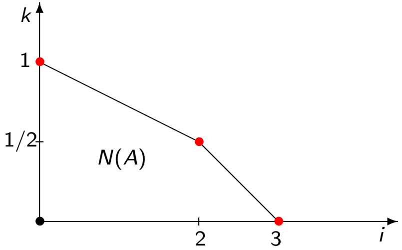
## Definition

Let $A(\xi',\lambda) = \sum_{\alpha,k} a_{\alpha k}\lambda^k(\xi')^\alpha$. Then the Newton polygon is defined as the convex hull of all points

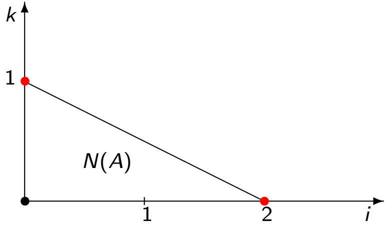
$$(|\alpha|, k) \quad \text{with} \quad a_{\alpha k} \neq 0$$

and their projections onto the axes.

(a) Stefan problem: $A(\xi', \lambda) = \lambda + |\xi'|^2 \sqrt{\lambda + |\xi'|^2}$.




(b) Heat equation: $A(\xi', \lambda) = \lambda + |\xi'|^2$.



# Definition of parabolicity

For standard equations $(\partial_t - A(D'))u = 0$, classical parabolicity is defined as

$$\lambda - A_0(\xi') \neq 0 \quad (\xi' \in \mathbb{R}^{n-1},\ \operatorname{Re}\lambda \geq 0,\ (\xi', \lambda) \neq 0).$$

Here, $A_0(\xi')$ is the principal symbol of $A(\xi')$.

In the Stefan problem we have the inhomogeneous symbol

$$A(\xi', \lambda) = \det L(\xi', \lambda) = \lambda + |\xi'|^2 \sqrt{|\xi'|^2 + \lambda}.$$

- What is the principal symbol?

# A family of principal symbols

What is the principal symbol of $A(\xi',\lambda) = \lambda + |\xi'|^2\sqrt{|\xi'|^2+\lambda}$?

Idea: For every $r>0$ we set $|\lambda| \approx |\xi'|^r$ and get a family of principal symbols $(A_r(\xi',\lambda))_{r>0}$:

| | |
|---|---|
| $0<r<2$: | $A_r = |\xi'|^3$, |
| $r=2$: | $A_r = |\xi'|^2\sqrt{\lambda+|\xi'|^2}$, |
| $2<r<4$: | $A_r = |\xi'|^2\sqrt{\lambda}$, |
| $r=4$: | $A_r = \lambda + |\xi'|^2\sqrt{\lambda}$, |
| $r>4$: | $A_r = \lambda$. |

# A family of principal symbols

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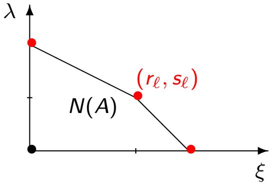
| $r = 4$: | $A_r = \lambda + |\xi'|^2 \sqrt{\lambda}$, |
| $r > 4$: | $A_r = \lambda$. |

## Definition (Gindikin-Volevich (1992), Mennicken-Volevich-D. (1998))

The scalar operator $A(D_{x'}, \partial_t)$ is called N-parabolic if for every $r > 0$ we have

$$A_r(\xi', \lambda) \neq 0 \quad (\operatorname{Re} \lambda \geq 0,\ \xi' \neq 0,\ \lambda \neq 0).$$

# Spaces related to the Newton polygon




For each vertex $(r_\ell, s_\ell)$ of the Newton polygon, we consider the space

$$\mathcal{F}_\ell^{s_\ell}((0,T), \mathcal{K}_\ell^{r_\ell}(\mathbb{R}^{n-1}))$$

with $\mathcal{F}_\ell \in \{B_{p_0 q_0}, H_{p_0}\}$, $\mathcal{K}_\ell \in \{B_{p_1 q_1}, H_{p_1}\}$, $p_i, q_i \in (1, \infty)$.

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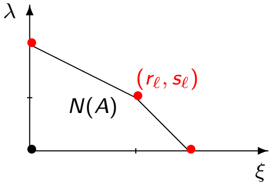


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The Sobolev space related to the Newton polygon $N(P)$ is the intersection of these spaces:

$$\mathbb{H} := \bigcap_\ell \mathcal{F}_\ell^{s_\ell}((0,T), \mathcal{K}_\ell^{r_\ell}(\mathbb{R}^n)).$$

- mixture of Bessel and Besov spaces can be chosen

# Main results

For scalar equations, we get (see also Gindikin-Volevich 1992, D.-Saal-Seiler 2008):

**Theorem (Kaip-D. 2013)**

*Let $A(\xi', \lambda)$ be N-parabolic. Then $A(D_{x'}, \partial_t)$ is an isomorphism in the spaces related to the Newton polygon $N(A)$.*

The proof uses the joint $H^\infty$-calculus of sectorial and bisectorial operators (Dore-Venni 2005) applied to $(\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \partial_t)$.

# N-parabolic systems

For systems, one has to consider the determinant:

**Theorem (Kaip-D. 2013)**

*Let* $\mathscr{L} = \big(\mathscr{L}_{jk}(\xi', \lambda)\big)_{j,k=1,\dots,N}$ *be a mixed-order matrix of symbols. Assume that* $\det \mathscr{L}$ *is N-parabolic. Then* $\mathscr{L}(D_{x'}, \partial_t)$ *is an isomorphism*

$$\mathscr{L}(D_{x'}, \partial_t) \in L_{Isom}\Big(\prod_{j=1}^{N} \mathbb{H}_j, \prod_{j=1}^{N} \mathbb{F}_j\Big),$$

*where the spaces are defined by the Newton polygon structure of the matrix.*

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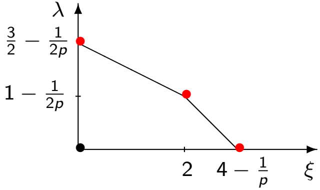
# Maximal regularity for the Stefan problem

We can apply the above result to the Stefan problem. The linearized problem is given by

$$\begin{aligned} (\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ u\big|_{\mathbb{R}^{n-1}} + \Delta' \sigma &= g \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ -\partial_{x_n} u\big|_{\mathbb{R}^{n-1}} + \partial_t \sigma &= h \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}. \end{aligned}$$

The spaces for $f, g, h$, and $u$ are standard. For $\sigma$ we have a Newton polygon space:



# Maximal regularity for the Stefan problem

## Theorem (Escher-Prüss-Simonett 2003, Volevich-D. 2008, Kaip-D. 2013)

*For the linearized Stefan problem we have maximal regularity in the following spaces, where $J = (0, T)$:*

$$\begin{aligned}
&f \in L^p(J; L^p(G)),\\
&g \in B_{pp}^{1-1/2p}(J; L^p(\partial G)) \cap L^p(J; B_{pp}^{2-1/p}(\partial G)),\\
&h \in B_{pp}^{1/2-1/2p}(J : L^p(\partial G)) \cap L^p(J; B_{pp}^{1-1/p}(\partial G)),\\
&u \in W_p^1(J; L^p(G)) \cap L^p(J; W_p^2(G)),\\
&\sigma \in B_{pp}^{3/2-1/2p}(J; L^p(\partial G)) \cap B_{pp}^{1-1/2p}(J; W_p^2(\partial G)) \cap L^p(J; B_{pp}^{4-1/p}(\partial G)).
\end{aligned}$$

# Further applications

The following problems are covered by the N-parabolic theory:

- Generalized thermoelastic plate equation in $\mathbb{R}^n$ (D.-Racke 2006),
- Generalized Stokes problem in $\mathbb{R}^n$ (Bothe-Prüß 2007),
- Spin-coating process (D.-Geissert-Hieber-Saal-Sawada 2011),
- Two-phase Navier-Stokes equation with surface tension and gravity (Prüß-Simonett 2009-2011), (Shibata-Shimizu 2011)
- Two-phase Navier Stokes equation with Boussinesq-Scriven surface fluid (Prüß-Bothe 2010),
- Fluid-structure interaction (D.-Saal, ongoing)

# Final remarks

- All results also hold in the $L^p$-$L^q$-setting where now Triebel-Lizorkin spaces appear (Kaip-D. 2013).
- All results have been obtained in the vector-valued case (with values in a UMD space).
- We considered the Newton polygon either in the whole space case or on the boundary after reduction to the boundary.
- The question of general mixed-order systems with mixed-order boundary conditions is still open.
  (see Faierman 2012, Faierman-D. 2012, Dreher-D. 2011 for partial results)
- There are also a priori estimates for elliptic-parabolic problems (Seger-D. 2016).

# Thank you for your attention!