Recent Progress in Coorbit Theory

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(joint work with F. De Mari, E. De Vito, G. Steidl, G. Teschke, and S. Vigogna)

Outline

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Classical Problem:

- analyze/decompose/approximate... a given signal/function
- First step: decomposition into suitable building blocks, transformation

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

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- Extract information of interest
- Reconstruct

Group Theory:

Common thread: square integrable group representations!

Definition

Let G be a l.c. topological group, π a unitary, irreducible represenation in a Hilbert space $\mathcal{H}$. $\psi \in \mathcal{H}$ is admisible, if

$$\int_G |\langle \psi, \pi(g)\psi \rangle|^2 dg = C_\psi < \infty$$

- well-defined voice transform

$$V_\psi : \mathcal{H} \longrightarrow L_2(G), \quad V_\psi(f)(g) := \langle f, \pi(g)\psi \rangle$$

- reproducing kernel property:

$$V_\psi(f) * K = V_\psi(f), \qquad K := V_\psi(\psi)$$

- inversion formula:

$$f = V_\psi^* V_\psi(f) = C_\psi^{-1} \int_G \langle f, \pi(g)\psi \rangle \pi(g)\psi dg$$

Examples:

- Wavelet transform $\leftrightarrow$ square-integrable representation of the affine group

$$\pi(a,x)f(t) = |a|^{-1/2} f\left(\frac{t-x}{a}\right)$$

- Gabor transform $\leftrightarrow$ representation of the Weyl–Heisenberg group

$$\pi(x,\omega,\tau)f(t) = e^{2\pi i\tau} e^{-i\pi x\omega} e^{2\pi i\omega t} f(t-x)$$

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- Idea: discretize the representation!

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- A unified treatment of the various different transforms.
- Natural families of smoothness spaces, smoothness measured by the decay of the voice transform.
- Frames for the scales of associated coorbit spaces, even Banach frames.

General Setting:

G group, π unitary representation in $\mathcal{H}$, weight function w i.e., $w(g \circ h) \leq w(g)w(h)$, $g, h \in G$

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$$\mathcal{A}_w := \{\psi : \int_G |\langle \psi, \pi(g)\psi \rangle| w(g) dg < \infty\} \text{ is non-trivial}$$

$$L_{p,w}(G) := \{f : \left(\int_G |f(g)|^p \, w(g)^p dg \right)^{1/p} < \infty\}$$

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$$\mathcal{H}_{1,w} := \{f \in \mathcal{H} : V_\psi(f) = \langle f, \pi(\cdot)\psi \rangle \in L_{1,w}(G)\}$$

$$\|f\|_{\mathcal{H}_{1,w}} := \|V_\psi f\|_{L_{1,w}}$$

Coorbit Spaces:

$$\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{1,w}^{\sim}, \qquad \mathcal{H}_{1,w}^{\sim} := \text{anti-dual}$$

$$\psi \in \mathcal{A}_w \Longrightarrow \text{extension to} \quad \mathcal{H}_{1,w}^{\sim} \quad \text{possible:}$$

$$V_\psi(f)(g) := \langle f, \pi(g)\psi \rangle_{\mathcal{H}_{1,w}^{\sim} \times \mathcal{H}_{1,w}}$$

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coorbit spaces:

$m(g \circ h \circ k) \leq w(g)m(h)w(k)$ moderate function

$$\mathrm{Co}(L_{p,m}) := \mathcal{H}_{p,m} := \{f \in \mathcal{H}_{1,w}^{\sim} : V_\psi(f) \in L_{p,m}(G)\},$$

$$\|f\|_{\mathcal{H}_{p,m}} := \|V_\psi f\|_{L_{p,m}(G)}$$

Classical Cases:

Wavelet transform: affine group

$$m(a, b) = |a|^{-s} \quad \rightsquigarrow$$

$$\mathcal{H}_{p,m} = \dot{B}^{s-1/2-1/p}_{p,p}$$ Besov spaces

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Gabor transform: Weyl-Heisenberg group

$$m(q, \omega) = (1 + |\omega|)^{2s} \quad \rightsquigarrow$$

$$\mathcal{H}_{p,m} = M^{s}_{pp} \qquad \text{modulation spaces}$$

Discretization:

Idea: find a discrete set $X = (g_\lambda)_{\lambda \in \Lambda}$ in G such that

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f) \pi(g_\lambda) \psi$$

- $e \in Q \subset G \quad X = (g_\lambda)_{\lambda \in \Lambda}$ is *Q*-dense if $\bigcup_{\lambda \in \Lambda} g_\lambda Q = G$

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- $$\begin{aligned} \mathrm{osc}_Q(g) &:= \sup_{u \in Q} |V_\psi(\psi)(ug) - V_\psi(\psi)(g)| \\ &= \sup_{u \in Q} |K(ug) - K(g)| \end{aligned}$$

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- $$\mathcal{B}_w := \{\psi \in L_2(\mathbb{R}^n) : V_\psi(\psi) \in \mathcal{W}(C_0, L_{1,w})\},$$
  $$\mathcal{W}(C_0, L_{1,w}) := \{F : \|(L_x \chi_{\mathcal{Q}})F\|_\infty \in L_{1,w}\}$$
  $$\|(L_x \chi_{\mathcal{Q}})F\|_\infty = \sup_{y \in x\mathcal{Q}} |F(y)|$$

Atomic Decomposition:

Theorem (Feichtinger/Gröchenig)

Suppose that $\mathcal{B}_w$ is nonempty. Choose Q so small that

$$\|osc_Q\|_{L_{1,w}(G)} < 1.$$

Then

(i) *Any $f \in \mathcal{H}_{p,m}$, $1 \leq p \leq \infty$, has an expansion*

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f)\pi(g_\lambda)\psi$$

where

$$C\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}} \leq \|f\|_{\mathcal{H}_{p,m}}.$$

(ii) *If $(d_\lambda)_{\lambda\in\Lambda} \in \ell_{p,m}$, then $f = \sum_{\lambda\in\Lambda} d_\lambda \pi(g_\lambda)\psi \in \mathcal{H}_{p,m}$*

$$\|f\|_{\mathcal{H}_{p,m}} \leq C'\|(d_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}}$$

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- Very important tool: Corresponding Principle

$$\mathcal{M}^{L_{p,m}} := \{F \in L_{p,m} \mid F * K = F\}$$

Theorem (Feichtinger/Gröchenig)

V_ψ induces an isomorhpism

$$V_\psi \; : \; \mathcal{H}_{p,m} \longleftrightarrow \mathcal{M}^{L_{p,m}}$$

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π not integrable: When does this happen?

- Paley–Wiener space

$$\mathcal{H} = B^2_\Omega = \{f \in L_2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq [-\omega, \omega]\}$$

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- [D./F. De Mari/E. De Vito/ D. Labate/ G. Steidl/ G. Teschke/S. Vigogna (2017)] Generalized coorbit theory can be established!

Test Functions:

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Theorem

Technical conditions $\Longrightarrow$ $\mathcal{S}$ is a π-invariant Fréchet space, $i : \mathcal{S} \longrightarrow \mathcal{H}$ is continuous, dense range, $i' : \mathcal{H} \longrightarrow \mathcal{S}'$ continuous, injective, dense range.

Distributions:

- Gelfand triple

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$$\mathcal{S} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}'$$

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- extended voice transform

$$\begin{array}{rcccl} V_\psi & : & \mathcal{S}' & \longrightarrow & C(G) \\ & & f & \longmapsto & \langle f, \pi(\cdot)\psi \rangle_{\mathcal{S}' \times \mathcal{S}} \end{array}$$

Coorbit Spaces:

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Theorem (Correspondence Principle)

$\mathrm{Co}(Y)$ is a π-invariant Banach space, V_ψ is an isometry from $\mathrm{Co}(Y)$ onto $\mathcal{M}^Y$,

$$V_\psi(\mathrm{Co}(Y)) = \mathcal{M}^Y$$

Typical Model for the Target Space:

- $I = (1, +\infty)$ target space:

$$\mathcal{T}_w = \bigcap_{p \in I} L_{p,w}(G)$$

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$$\mathcal{U}_w = \text{span} \bigcup_{q \in I} L_{q,w^{-1}}(G),$$

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Theorem

$\mathcal{T}_w$ is a reflexive Fréchet space, $\mathcal{U}_w$ is reflexive, locally compact topological vector space, and

$$\mathcal{T}_w' = \mathcal{U}_w$$

Generalizes results of [Davis/Murray/Weber (1970)]

It works....

Theorem

Suppose that there exists $\psi \in \mathcal{H}$ *s.t.* $K \in L_{p,w}(G)\ \forall\ p \in I$
Set

$$\mathcal{S}_w : \{f \in \mathcal{H} \mid \langle f, \pi(\cdot)\psi\rangle_{\mathcal{H}} \in L_{p,w}(G)\ \forall p \in I\}$$

Then

i) $\mathcal{S}_w$ *is an invariant, reflexive Fréchet space*

ii) *the extended voice transform is continuous, injective into* $\mathcal{U}_w$*, and*

$$\begin{aligned} range(V_\psi) &= \mathcal{M}^{\mathcal{U}_w} := \{\Phi \in \mathcal{U}_w \mid \Phi * K = \Phi\} \\ &= span \bigcup_{p\in I} \mathcal{M}^{L_{p,w}(G)} \subset L_{\infty,w^{-1}}(G) \end{aligned}$$

+several additional properties

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Theorem

It holds

i) $\psi \in B^2_\Omega$ *is admissible* $\iff \quad |\widehat{\psi}| = 1$ *a.e.* Ω.

$$K = \langle \psi, \pi(\cdot)\psi\rangle_{\mathcal{H}} = \mathcal{F}^{-1}\chi_\Omega,$$

ii) *If* $\Omega = [-\omega, \omega]$

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- target space:

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- distributions:

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Theorem

The extended voice transform is the inclusion

$$V : \mathcal{S}' \hookrightarrow \mathcal{U}$$

and the coorbits of the L_p spaces are

$$\mathrm{Co}(L_p(\mathbb{R})) = \mathcal{M}^{L_p} = B^p_\Omega.$$

Schrödingerlets:

- the group:

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- G can be realized as the triangular subgroup of $Sp(2,\mathbb{R})$

$$\begin{bmatrix} a^{-1/2}R & 0 \\ ba^{-1/2}R & a^{1/2}R \end{bmatrix}, \; R \in SO(2), R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix},$$

- π is equivalent to the metaplectic representation μ

$$\mu(b,a,\varphi)f(\xi) = a^{1/2}e^{-2\pi i|\xi|^2} f(a^{1/2}R_{-\varphi}\xi), \quad f \in L_2(\widehat{\mathbb{R}^2}).$$

Schrödingerlets:

- $\widehat{\mu}(g)f = \mathcal{F}^{-1} \circ \mu(g) \circ \mathcal{F}, \quad b \mathrel{\hat{=}} \text{time} \Longrightarrow$

$$(b,x) \mapsto \widehat{\mu}_b f(x) = \widehat{\mu}(b,1,0)f(x) = \int_{\widehat{\mathbb{R}^2}} \hat{f}(\xi) e^{-2\pi i b|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi.$$

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- flow satisfies the Schrödinger equation:

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Theorem

There exist admissible vectors $\psi \in \mathcal{H}$ whose kernel $K = V_\psi(\psi)$ belongs to $\bigcap_{p \in I} L_p(G)$ but not to $L_1(G)$.

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- Related work: [J.G. Christensen, G. Ólafsson (2009), (2011)]

Discretization:

Classical approach does not carry over! (Based on a Neumann series argument etc.)

$$\mathcal{T}_w = \bigcap_{p \in I} L_{p,w}(G), \ \mathcal{U}_w = \operatorname{span} \bigcup_{q \in I} L_{q,w^{-1}}(G), \ \ Y = L_{p,m}$$

Basic assumptions (BA):

- X_n sequence Q_{ϵ_n}–dense sets where $\lim_{n\to\infty} Q_{\epsilon_n} = \{e\}$,

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- $X_{m(n)} \subset X_n, \#X_{m(n)} = m$ s.t. $\cup_{n=1}^{\infty} X_{m(n)}$ is dense in G,
- for some r satisying $1/q + 1/r = 1 + 1/p$

$$K \in L_{r,m^{p/r}}$$

$$\tilde{K}(x) := \sup_{u\in Q} |K(ux)| \in L_{r,m^{p/r}}$$

$$\operatorname{osc}_Q(K) \in L_{r,m^{p/r}}$$

The Result:

Theorem

BA satisfied $\Longrightarrow$

i) *For every* $f \in \mathrm{Co}(L_{p,m})$ *and any given* $\epsilon > 0$ $\exists X_{m(n(\epsilon))}$ *and coefficients* $(c(f)_{n,g})_{g \in X_{m(n(\epsilon))}}$ *s. t.*

$$\|f - \sum_{g \in X_{m(n(\epsilon))}} c(f)_{n,g}\pi(g)\psi\|_{\mathrm{Co}(L_{p,m})} \leq C_1\epsilon$$

where

$$\|(c(f)_{n,g})_{g \in X_{m(n(\epsilon))}}\|_{\ell_{q,m^{p/q}}(X_{m(n(\epsilon))})} \leq C(n)\|f\|_{\mathrm{Co}(L_{p,m})}$$

ii) *Conversely, if* $(d_g)_{g \in X_n} \in \ell_{q,m^{p/q}}$, *then* $f = \sum_{g \in X_n} d_g \pi(g)\psi$ *is in* $\mathrm{Co}(L_{p,m})$ *and*

$$\|f\|_{\mathrm{Co}(L_{p,m})} \leq C_2\|\mathrm{osc}_Q(K) + |K|\|_{L_{r,m^{p/r}}}\|(d_g)_{g \in X_n}\|_{\ell_{q,m^{p/q}}}.$$

Idea of Proof:

- Partition of unity:

$$\text{supp}\phi_g \subseteq gQ, \quad 0 \le \phi_g \le 1 \text{ for all } g \in X_n, \quad \sum_{g \in X_n} \phi_g = 1$$

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- Partition of unity:

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- Show that the operator

$$T_{m(n)}F := \sum_{g \in X_{m(n)}} \langle F, \phi_g \rangle L_g K \text{ on } \mathcal{M}^{L_{qm^{p/q}}}$$

has a right inverse $S_{m(n)}, \Longrightarrow$

$$F = T_{m(n)} \circ S_{m(n)}(F) = \sum_{g \in X_{m(n)}} \langle S_{m(n)} F \phi_g \rangle L_g K$$

Idea of Proof:

- Corresponding principle:

$$f = V_\psi^{-1} \circ V_\psi(f) = \sum_{g \in X_{m(n)}} \langle S_{m(n)} \circ V_\psi(f), \phi_g \rangle \pi(g)\psi$$

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- Density arguments etc. $\Longrightarrow \square$
- Problem: no uniform bound for $\|S_{m(n)}\|$!

An Example:

Once again: the band–limited case!

- $Q_n = [-\frac{1}{2n}, \frac{1}{2n}], X_n := \{\frac{i}{n}, n \in \mathbb{Z}\}$
  $\phi_g = \chi_{Q_n}(\cdot - g), \; X_{m(n)} = \{i \;:\; -n^2 \leq i \leq n^2\}$

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- Eigenvalues not uniformly bounded from below! (A. Böttcher, privat communication)

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- If the representation is not integrable: Use Fréchet–spaces as target and test spaces.
- Coorbit spaces related with band–limited functions.
- Discretization is complicated!

Thanks a lot for your attention!

References:

1 J.G. Christensen and G. Olafsson, *Coorbit spaces for dual pairs,* Appl. Comput. Harmon. Anal. **31** (2011), 303–324.

2 S. Dahlke, F. De Mari, E. De Vito, D. Labate, G. Steidl, G. Teschke, and S. Vigogna, *Coorbit spaces with voice in a Fréchet space*, J. Fourier Anal. Appl. **23** (2017), 141–206. .

3 S. Dahlke, F. De Mari, E. De Vito, S. Häuser, G. Steidl, and G. Teschke, *Different faces of shearlet groups,* J. Geom. Anal. **26** (2016), 1693-1729.

4 S. Dahlke, G. Steidl, and G. Teschke, *The continuous shearlet transform in arbitrary space dimensions,* J. Fourier Anal. Appl. **16** (2010), 340–364.

5 S. Dahlke, G. Steidl, and G. Teschke, *Shearlet coorbit spaces: Compactly supported analyzing shearlets, traces and embeddings,* J. Fourier Anal. Appl. **17** (2011), 1232–1255.

6 H. G. Feichtinger and K. Gröchenig, *A unified approach to atomic decompositions via integrable group representations*, Proc. Conf. "Function Spaces and Applications", Lund 1986, Lecture Notes in Math. **1302** (1988), 52–73.

7 H. G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decomposition I,* J. Funct. Anal. **86** (1989), 307–340.

Reconstruction:

Theorem (Feichtinger/Gröchenig)

$\mathcal{B}_w$ nonempty. Choose Q such that

$$\|osc_Q\|_{L_{1,w}(G)} < 1/\|V_\psi(\psi)\|_{L_{1,w}(G)}.$$

Then the set $\{\pi(g_\lambda)\psi : \lambda \in \Lambda\}$ is a Banach frame for $\mathcal{H}_{p,m}$. This means that

(i) $f \in \mathcal{H}_{p,m}$ *iff* $(\langle f, \pi(g_\lambda)\psi\rangle_{\mathcal{H}^{\sim}_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda} \in \ell_{p,m}$;

(ii)

$$\|f\|_{\mathcal{H}_{p,m}} \sim \|(\langle f, \pi(g_\lambda)\psi\rangle_{\mathcal{H}^{\sim}_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\|_{\ell_{p,m}}$$

(iii) *there exists a bounded, linear reconstruction operator $\mathcal{S}$ from $\ell_{p,m}$ to $\mathcal{H}_{p,m}$ such that*
$$\mathcal{S}\left((\langle f, \pi(g_\lambda)\psi\rangle_{\mathcal{H}^{\sim}_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\right) = f.$$