Sparse approximation with respect to the Faber-Schauder system

Joint work with Tino Ullrich

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Bedlewo, September 2017

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Motivation: Nonlinear Approximation

$$S_m f = \sum_{|k|<m} \hat{f}(k) e^{ikx}$$

- $f(x) = |sin(x)|$

$$\|f - S_m f\|_\infty \asymp m^{-1}$$

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- Trigonometric system: $\{e^{ikx} \ : \ k \in \mathbb{Z}\}$
- **Majorov** (1970s): $|\Lambda_m| = m$

$$\left\| f - \sum_{k \in \Lambda_m} \lambda_k e^{ikx} \right\|_\infty \asymp m^{-3/2}$$

(Improves on a result by **Ismagilov**: $m^{-6/5+\varepsilon}$)

- Constructive approach!

Function spaces with bounded mixed derivatives - Sobolev

- $f : \mathbb{R}^d \to \mathbb{C}$, multivariate
- r **smoothness**, $1 < p < \infty$ **integrability**
- r integer

$$\|f|S_p^r W(\mathbb{R}^d)\| := \sum_{|\boldsymbol{\alpha}|_\infty \leq r} \|D^{\boldsymbol{\alpha}} f\|_{L_p(\mathbb{R}^d)}$$

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- $d = 2$

$$\|f\|_{S^r_pW(\mathbb{R}^2)} := \|f\|_{L_p(\mathbb{R}^2)} + \left\|\frac{\partial^r f}{\partial x_1^r}\right\|_{L_p(\mathbb{R}^2)} + \left\|\frac{\partial^r f}{\partial x_2^r}\right\|_{L_p(\mathbb{R}^2)} + \left\|\frac{\partial^{2r} f}{\partial x_1^r \partial x_2^r}\right\|_{L_p(\mathbb{R}^2)}$$

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- $r > \frac{1}{p}$

$$S^r_pW(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$$

Function spaces with bounded mixed differences - Besov

- Iterated differences

$$\Delta_h^m f(x) := \begin{cases} f(x+h) - f(x) & : \quad m = 1 \\ \Delta_h^1 \Delta_h^{m-1} f(x) & : \quad \text{otherwise.} \end{cases}$$

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- The Besov norm in case $d = 2$, **fine index** $0 < \theta \leq \infty$

$$\|f\|_{S^r_{p,\theta}B(\mathbb{R}^2)} := \|f\|_{L_p(\mathbb{R}^2)} + \Big(\sum_{j_1=0}^{\infty} 2^{rj_1\theta} \sup_{|h| \leq 2^{-j_1}} \|\Delta^m_{h,1} f\|^\theta_{L_p(\mathbb{R}^2)} \Big)^{1/\theta}$$

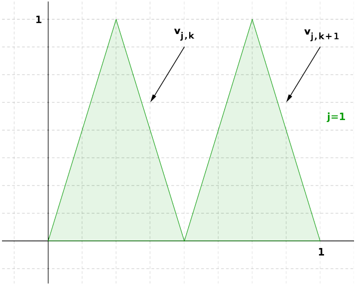
$$+ \Big(\sum_{j_2=0}^{\infty} 2^{rj_2\theta} \sup_{|h| \leq 2^{-j_2}} \|\Delta^m_{h,2} f\|^\theta_{L_p(\mathbb{R}^2)} \Big)^{1/\theta}$$

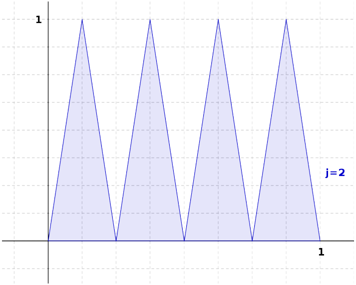
$$+ \Big(\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 2^{r(j_1+j_2)\theta} \sup_{\substack{|h_1| \leq 2^{-j_1} \\ |h_2| \leq 2^{-j_2}}} \|\Delta^m_{h_1,1} \Delta^m_{h_2,2} f\|^\theta_{L_p(\mathbb{R}^2)} \Big)^{1/\theta}$$

The Faber-Schauder basis

- **Faber 1908:** Univariate hat functions: decomposition of $f \in C([0,1])$

$$f(x) = f(0)v_{-1,0} + f(1)v_{-1,1} + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}(f)v_{j,k}$$





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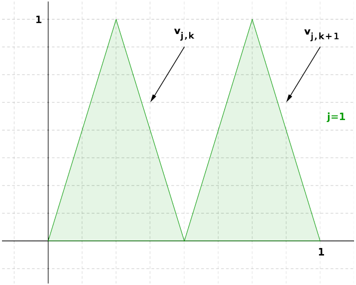
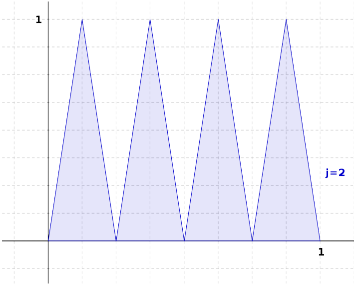
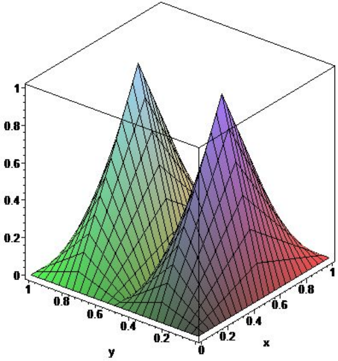


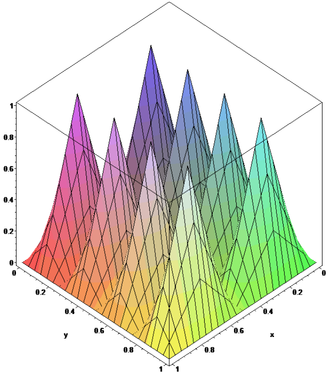


coefficients: **function evaluations**

The tensorized Faber-Schauder basis

$d > 1? \implies$ **tensorize** the univariate hat functions





$$f \in C([0,1]^d) \implies f = \sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^d} \sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} d_{\boldsymbol{j},\boldsymbol{k}} v_{\boldsymbol{j},\boldsymbol{k}}, \quad |D_{\boldsymbol{j}}| \asymp 2^{|\boldsymbol{j}|_1}$$

Characterization via Faber series - Sobolev case

Theorem (B., Ullrich)

Let $1 < p < \infty$ *and* $\max\{1/p, 1/2\} < r < 1 + \min\{1/p, 1/2\}$. *Then every* $f \in S^r_p W([0,1]^d)$ *can be represented by*

$$f = \sum_{\boldsymbol{j} \in \mathbb{N}^d_{-1}} \sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} d_{\boldsymbol{j},\boldsymbol{k}}(f) v_{\boldsymbol{j},\boldsymbol{k}}$$

with uncond. convergence in $S^r_p W([0,1]^d)$. *Furthermore we have*

$$\begin{aligned} \|f\|_{S^r_p W([0,1]^d)} &\asymp \Big\| \Big(\sum_{\boldsymbol{j} \in \mathbb{N}^d_0} 2^{2r|\boldsymbol{j}|_1} \Big| \sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} d_{\boldsymbol{j},\boldsymbol{k}}(f) \chi_{\boldsymbol{j},\boldsymbol{k}}(\cdot) \Big|^2 \Big)^{1/2} \Big\|_p \\ &:= \|d_{\boldsymbol{j},\boldsymbol{k}}(f) | s^{r,\Omega}_p w\|. \end{aligned}$$

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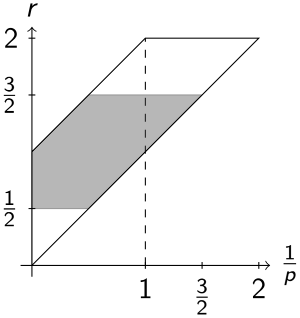
Theorem (B., Ullrich)

Let $1 < p < \infty$ *and* $\max\{1/p, 1/2\} < r < 2$. *Then we have*

$$\|d_{\boldsymbol{j},\boldsymbol{k}}(f)|s_p^{r,\Omega}w\| \lesssim \|f\|_{S_p^rW([0,1]^d)}$$



$\frac{1}{p} < r < \frac{1}{2}$, $p > 2$: **no** norm equivalence, Haar system, **Seeger/Ullrich** '15

Characterization via Faber series - Sobolev case

Similar results for **Besov** spaces: $1/2 < p \leq \infty$, $0 < \theta < \infty$ and $1/p < r < \max\{1 + 1/p, 2\}$

$$\|f\|_{S^r_{p,\theta}B([0,1]^d)} \asymp \|d_{\boldsymbol{j},\boldsymbol{k}}(f)|s^{r,\Omega}_{p,\theta}b\| := \Big(\sum_{\boldsymbol{j}\in\mathbb{N}^d_{-1}} 2^{\theta|\boldsymbol{j}|_1(r-\frac{1}{p})}\Big(\sum_{\boldsymbol{k}\in D_{\boldsymbol{j}}} |d_{\boldsymbol{j},\boldsymbol{k}}|^p\Big)^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}.$$

Triebel (2010)

Sparse approximation

- Best m-term approximation wrt. a dictionary $\mathcal{D} = \{\varphi_j\}_j \subset X$
- X quasi-Banach space, $f \in X$

$$\sigma_m(f, \mathcal{D})_X = \inf \left\{ \left\| f - \sum_{j \in \Lambda} \lambda_j \varphi_j \right\|_X \ : \ |\Lambda| \leq m, \lambda_j \in \mathbb{C}, \varphi_j \in \mathcal{D} \right\}$$

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- Best m-term widths

$$\sigma_m(\mathbf{F},\mathcal{D})_X := \sup_{f\in\mathbf{F}} \sigma_m(f,\mathcal{D})_X$$

- Let $\mathcal{F}^d := (v_{\boldsymbol{j},\boldsymbol{k}})_{\boldsymbol{j}\in\mathbb{N}^d_{-1},\boldsymbol{k}\in D_{\boldsymbol{j}}}$ be the Faber-Schauder dictionary

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- we are interested in $\sigma_m(S_p^r W([0,1]^d), \mathcal{F}^d)_{L_q([0,1]^d)}$
- representation theorem: transfer the problem from **function spaces** $S_p^r W \hookrightarrow L_q$ to **sequence spaces** $s_{p,\Omega}^r w \hookrightarrow s_{q,1}^{0,\Omega} b$

Sparse Faber-Schauder approximation

The quantities $\sigma_m(s_p^{r,\Omega}w, \mathbb{E}^d)_{s_{q,1}^{0,\Omega}b}$ were studied in:

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Theorem

1. *Let* $1 < p < q \leq \infty$ *and* $\max\{1/p, 1/2\} < r < 2$ *then*

$$\sigma_m(S_p^r W([0,1]^d), \mathcal{F}^d)_{L_q([0,1]^d)} \lesssim m^{-r}(\log m)^{(d-1)(r+1/2)}.$$

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2. *Let* $1 < p < q \leq \infty$, $0 < \theta \leq \infty$ *and* $\max\{\frac{1}{p}, \frac{1}{\theta} - \frac{1}{q}\} < r < 2$ *then*

$$\sigma_m(S_{p,\theta}^r B([0,1]^d), \mathcal{F}^d)_{L_q([0,1]^d)} \lesssim m^{-r}(\log m)^{(d-1)(r+1-\frac{1}{\theta})}.$$

Besov spaces and small smoothness

We consider the case $0 < \theta < 1$ and consider the embedding

$$S^r_{p,\theta}B([0,1]^d) \hookrightarrow L_q([0,1]^d)$$

with $\frac{1}{p} - \frac{1}{q} < r \leq \frac{1}{\theta} - 1$

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- smaller fine index in the model space than in the target space
- model space: quasi-Banach space

Besov spaces and small smoothness

Theorem (B., Ullrich)

Let $1 \leq p < q \leq \infty$, $0 < \theta < 1$ *and* $\frac{1}{p} < r < \min\left\{\frac{1}{\theta} - 1, 2\right\}$ *or* $\frac{1}{p} < r = \frac{1}{\theta} - 1 < 2$ *then*

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$\boldsymbol{q = \infty}$: As far as we know: **first** sharp result for best m-term approximation in mixed spaces with target space $L_\infty([0,1]^d)$

The constructive algorithm...part 1

Put all levels of the same order $|\boldsymbol{j}|_1 = \mu$ together:

$$\sigma_m(f, \mathcal{F}^d)_{L_q} \leq \inf_{"\lambda"} \sum_{\boldsymbol{j} \in N_0^d} 2^{-\frac{|\boldsymbol{j}|_1}{q}} \Big(\sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} |d_{\boldsymbol{j},\boldsymbol{k}}(f) - \lambda_{\boldsymbol{j},\boldsymbol{k}}|^q \Big)^{\frac{1}{q}}$$

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Put all levels of the same order $|\boldsymbol{j}|_1 = \mu$ together:

$$\begin{aligned} \sigma_m(f, \mathcal{F}^d)_{L_q} &\leq \inf_{"\lambda"} \sum_{\boldsymbol{j} \in N_0^d} 2^{-\frac{|\boldsymbol{j}|_1}{q}} \Big(\sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} |d_{\boldsymbol{j},\boldsymbol{k}}(f) - \lambda_{\boldsymbol{j},\boldsymbol{k}}|^q \Big)^{\frac{1}{q}} \\ &= \inf_{"\lambda"} \sum_{\mu=0}^{\infty} \sum_{|\boldsymbol{j}|_1 = \mu} 2^{-\frac{\mu}{q}} \Big(\sum_{\boldsymbol{k} \in D_{\boldsymbol{j}}} |\underbrace{d_{\boldsymbol{j},\boldsymbol{k}}(f) - \lambda_{\boldsymbol{j},\boldsymbol{k}}}_{=: a_{\boldsymbol{j},\boldsymbol{k}}}|^q \Big)^{\frac{1}{q}} \end{aligned}$$

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&= \inf_{''\lambda''} \sum_{\mu=0}^{\infty} \|A_\mu\|_{s^{0,\Omega}_{q,1} b}
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The constructive algorithms...part 2

We have to minimize $\sum_{\mu=0}^{\infty} \|A_\mu\|_{s^{0,\Omega}_{q,1}b}$ by spending m_μ nonzero $\lambda_{\boldsymbol{j},\boldsymbol{k}}$ terms to A_μ:

$$A_\mu = \begin{pmatrix} a_{\boldsymbol{j}^1,\boldsymbol{k}^1}(f) & \dots & a_{\boldsymbol{j}^1,\boldsymbol{k}^{2^\mu+1}}(f) \\ \cdot & & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \cdot \\ a_{\boldsymbol{j}^{\mu^{d-1}},\boldsymbol{k}^1}(f) & \dots & a_{\boldsymbol{j}^{\mu^{d-1}},\boldsymbol{k}^{2^\mu+1}}(f) \end{pmatrix}$$

The constructive algorithms...regular smoothness

$$\ldots \lesssim \sum_{\mu=0}^{\infty} \mu^{(d-1)(1-\frac{1}{\max\{q,1\}})} \|A_\mu\|_{s^{0,\Omega}_{\max\{q,1\},\max\{q,1\}}b}$$

Lemma (Pietsch, Temlyakov, Stechkin...)

Let $0 < p < q \leq \infty$, $m < n$

$$\sigma_m(\ell_p^n, \mathbb{E})_{\ell_q^n} \asymp m^{-(\frac{1}{p}-\frac{1}{q})}$$

Algorithm - small smoothness - part 1

Lemma (B., Ullrich)

Let $\frac{1}{p} - \frac{1}{q} < \frac{1}{\theta} - 1$ *then*

$$\sigma_m(\ell_\theta^b(\ell_p^d), \mathbb{E})_{\ell_\nu^b(\ell_q^d)} \lesssim \begin{cases} \left(\frac{1}{m}\right)^{\frac{1}{p}-\frac{1}{q}} & : \quad m < d \\ \left(\frac{1}{d}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{d}{m}\right)^{\frac{1}{\theta}-\frac{1}{\nu}} & : \quad d \leq m < bd \\ 0 & : \quad m \geq bd. \end{cases}$$

Idea:

$$\sigma_m(\ell_\theta^b(\ell_p^d), \mathbb{E})_{\ell_\nu^b(\ell_q^d)} \lesssim \sup_{0 \leq s \leq m} \sigma_s(\ell_p^d, \mathbb{E})_{\ell_q^d} \left(\frac{s}{m}\right)^{\frac{1}{\theta}-\frac{1}{\nu}},$$

that was motivated by result of **Edmunds/Netrusov** on **entropy numbers** of **vector valued spaces** .

Algorithm - small smoothness - part 2

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,d} \\ \cdot & & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \cdot \\ a_{b,1} & \dots & a_{b,d} \end{pmatrix}$$

Assuming $\|A|\ell_\theta^b(\ell_p^d)\| \leq 1.$ We spend to the i-th line

$$m_i = \|a_{i,\cdot}|\ell_p^d\|^\theta m$$

biggest zero coefficients.

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$\Longrightarrow$ strategy is a m-term approximation of the matrix A.

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$$m_i = \|a_{i,\cdot}|\ell_p^d\|^\theta m$$

biggest zero coefficients.Obviously,

$$\sum_{i=0}^{b-1} m_i = m \sum_{i=0}^{b-1} \|a_{i,\cdot}|\ell_p^d\|^\theta = m\|a|\ell_\theta^b(\ell_p^d)\|^\theta \leq m$$

$\Longrightarrow$ strategy is a m-term approximation of the matrix A.

Sparse approximation vs. linear sparse grid sampling

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$$\sigma_m(S^r_pW,\mathcal{F}^d)_q := \inf_{(x_j),(\varphi_j)} \sup_{\|f|S^r_pW\|\leq 1} \|f - \sum_{j=0}^{m} f(x_j)\varphi_j\|_q$$