# On estimates of convolutions in Morrey-type spaces

V.I. Burenkov

S.M. Nikol'skii Mathematical Institute

Peoples Friendshp University of Russia (RUDN University)

International Conference

"New Perspectives in the Theory of Function Spaces,

and their Applications (NPFSA 2017)"

Bedlewo, September 18, 2017

## General Morrey-type spaces

Let $B(x,r)$ be the open ball in $\mathbb{R}^n$ of radius $r>0$ centered at the point $x\in\mathbb{R}^n$.

**Definition 1.** *Let* $0<p,\theta\leqslant\infty$ *and let* $w$ *be a nonnegative Lebesgue measurable function on* $(0,\infty)$.

*The local Morrey-type space* $LM_{p\theta,w(\cdot)}\equiv LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ *is the space of all functions* $f$ *Lebesgue measurable on* $\mathbb{R}^n$ *with finite quasinorm*

$$\|f\|_{LM_{p\theta,w(\cdot)}}=\left\|w(r)\|f\|_{L_p(B(0,r))}\right\|_{L_\theta(0,\infty)}.$$

*The global Morrey-type space* $GM_{p\theta,w(\cdot)}\equiv GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ *is the space of all functions* $f$ *Lebesgue measurable on* $\mathbb{R}^n$ *with finite quasinorm*

$$\|f\|_{GM_{p\theta,w(\cdot)}}=\sup_{x\in\mathbb{R}^n}\|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}}=\sup_{x\in\mathbb{R}^n}\left\|w(r)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(0,\infty)}.$$

If $\theta=\infty$ и $w(\cdot)\equiv 1$, then $LM_{p\infty,1}=GM_{p\infty,1}=L_p(\mathbb{R}^n)$, and if $\theta=\infty$ и $w(\cdot)=r^{-\lambda}$, $0<\lambda\leqslant\frac{n}{p}$, then

$$GM_{p\infty,r^{-\lambda}}\equiv M_p^\lambda$$

is the classical Morrey space.

## Nontriviality conditions of Morrey-type spaces

**Definition 2.** *Let $0 < p, \theta \leqslant \infty$. $\Omega_\theta$ is the space of all functions $w$, which are nonnegative Lebesgue measurable on $(0,\infty)$, not equivalent to $0$ on $(t,\infty)$ for all $t > 0$ and such that for some $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{1}$$

*Moreover, $\Omega_{p\theta}$ is the space of all functions $w$, which are nonnegative Lebesgue measurable on $(0,\infty)$, not equivalent to $0$ on $(t,\infty)$ for all $t > 0$ and such that for some $t > 0$*

$$\left\|w(r) r^{\frac{n}{p}}\right\|_{L_\theta(0,t)} < \infty\,, \qquad \|w(r)\|_{L_\theta(t,\infty)} < \infty\,, \tag{2}$$

*or, which is equivalent,*

$$\left\|w(r)\left(\frac{r}{t+r}\right)^{\frac{n}{p}}\right\|_{L_\theta(0,\infty)} < \infty\,. \tag{3}$$

Note that if condition (2) (hence also condition (3)) is satisfied for some $t > 0$, then it is satisfied for all $t > 0$.

**Lemma.** *Let $0 < p, \theta \leqslant \infty$ and let $w$ be a nonnegative Lebesgue measurable function on $(0,\infty)$, not equivalent to $0$ on $(t,\infty)$ for all $t > 0$.*

*Then the space $LM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w \in \Omega_\theta$, and the space $GM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w \in \Omega_{p\theta}$.*

## Young's inequality for convolutions of functions

Let $f_1, f_2$ be Lebesgue measurable functions on $\mathbb{R}^n$ and

$$(f_1 * f_2)(x) = \int_{\mathbb{R}^n} f_1(x-y) f_2(y) dy \,, \quad x \in \mathbb{R}^n \,,$$

be the convolution of these functions.

The classical.Young's inequality for convolutions of functions for the Lebesgue spaces has the following form:

$$\|f_1 * f_2\|_{L_p} \leqslant \|f_1\|_{L_{p_1}} \|f_2\|_{L_{p_2}} \tag{4}$$

for all $f_k \in L_{p_k}, k = 1, 2$, where

$$1 \leqslant p_1, p_2 \leqslant p \leqslant \infty, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1 \,.$$

If $p_1 = p$, then it takes the form

$$\|f_1 * f_2\|_{L_p} \leqslant \|f_1\|_{L_p} \|f_2\|_{L_1} \,. \tag{5}$$

## An analogue of Young's inequality for convolutions of functions for general Morrey-type spaces

**Theorem 1.** *Let*

$$1 \leqslant p_1, p_2 \leqslant p \leqslant \infty, \quad p_1 \leqslant \theta_1 \leqslant \infty, \ p_2 \leqslant \theta_2 \leqslant \infty, \quad 0 \leqslant \alpha_1, \alpha_2 \leqslant 1 \tag{6}$$

*and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \qquad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{1}{p}, \qquad \frac{\alpha_1}{\theta_1} + \frac{\alpha_2}{\theta_2} = \frac{1}{\theta} \tag{7}$$

*Moreover, let* $w_1 \in \Omega_{p_1\theta_1}, w_2 \in \Omega_{p_2\theta_2}$ *u*

$$w(r) = w_1^{\alpha_1}(r) w_2^{\alpha_2}(r), \quad r > 0. \tag{8}$$

*Then* $w \in \Omega_{p\theta}$, *for all* $f_k \in GM_{p_k\theta_k,w_k(\cdot)} \cap L_{p_k}, k = 1, 2$, *the convolution* $f_1 * f_2$ *exists almost everywher on* $\mathbb{R}^n$ *and*

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \leqslant \|f_1\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}} \|f_2\|^{1-\alpha_2}_{L_{p_2}} . \tag{9}$$

## Particular cases

We note the following particular cases of inequality (9).

1) If $\alpha_1 = 1, \alpha_2 = 0, p_1 = p, p_2 = 1, \theta_1 = \theta, \theta_2 = \infty, w_1(\cdot) = w(\cdot), w_2(\cdot) \equiv 1$, then

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \leqslant \|f_1\|_{GM_{p\theta,w(\cdot)}} \|f_2\|_{L_1}. \tag{10}$$

In particular, for $1 \leqslant p \leqslant \infty, 0 \leqslant \lambda \leqslant \frac{n}{p}$

$$\|f_1 * f_2\|_{M_p^\lambda} \leqslant \|f_1\|_{M_p^\lambda} \|f_2\|_{L_1}.$$

These are direct analogues of Young's inequality for convolutions of functions ($L_p$ is replaced by $GM_{p\theta,w(\cdot)}$ , by $M_p^\lambda$ respectively).

Note that in these inequalities it is essential that the global Morrey-type spaces $GM_{p\theta,w(\cdot)}$ are used, in particular Morrey spaces $M_p^\lambda$. Namely, for any $A > 0$ the inequality

$$\|f_1 * f_2\|_{LM_p^\lambda} \leqslant A\|f_1\|_{LM_p^\lambda} \|f_2\|_{L_1}.$$

cannot hold for all $f_1 \in LM_p^\lambda, f_2 \in L_1$.

Also note that if $p_2 > 1$, then the direct analogues of the Young's inequality

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \leqslant A\|f_1\|_{GM_{p_1\theta,w(\cdot)}} \|f_2\|_{L_{p_2}}.$$

$\left(\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1\right)$, in particular,

$$\|f_1 * f_2\|_{M_p^\lambda} \leqslant A\|f_1\|_{M_{p_1}^\lambda} \|f_2\|_{L_{p_2}},$$

$(0 < \lambda < \frac{n}{p})$ cannot hold for any $A > 0$ independent of $f_1, f_2$.

2) If $\alpha_1 = \frac{p_1}{p}, \alpha_2 = 0, \theta_1 = \frac{p_1}{p}\theta, \theta_2 = \infty, \theta \geqslant p, w_1(\cdot) = w^{\frac{p}{p_1}}(\cdot), w_2(\cdot) = 1$, then

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \leqslant \|f_1\|^{\frac{p_1}{p}}_{GM_{p_1,\frac{p_1}{p}\theta,w^{\frac{p}{p_1}}(\cdot)}} \|f_1\|^{1-\frac{p_1}{p}}_{L_{p_1}} \|f_2\|_{L_{p_2}}. \tag{11}$$

3) If $\theta_1 = \theta_2 = \theta = \infty$, $0 \leqslant \lambda_1 \leqslant \frac{n}{p_1}, 0 \leqslant \lambda_2 \leqslant \frac{n}{p_2}$, $w_1(r) = r^{-\lambda_1}, w_2(r) = r^{-\lambda_2}, w(r) = r^{-(\alpha_1\lambda_1+\alpha_2\lambda_2)}$, then

$$\|f_1 * f_2\|_{M_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \leqslant \|f_1\|^{\alpha_1}_{M^{\lambda_1}_{p_1}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{M^{\lambda_2}_{p_2}} \|f_2\|^{1-\alpha_2}_{L_{p_2}}. \tag{12}$$

For fixed $p_1, p_2, \lambda_1, \lambda_2$ by this inequality the maximal value of the parameter $\lambda$, for which $f_1 * f_2 \in M_p^\lambda$ is equal to $\max\{\frac{p_1\lambda_1}{p}, \frac{p_2\lambda_2}{p}\}$.

4) If $\theta_1 = \theta_2 = \theta = \infty, \alpha_1 = \frac{p_1}{p}, \alpha_2 = 0, w_1(r) = r^{-\lambda_1}, w_2(r) = 1, w(r) = r^{-\frac{p_1}{p}\lambda_1}$, then

$$\|f_1 * f_2\|_{M_p^{\frac{p_1}{p}\lambda_1}} \leqslant \|f_1\|^{\frac{p_1}{p}}_{M^{\lambda_1}_{p_1}} \|f_1\|^{1-\frac{p_1}{p}}_{L_{p_1}} \|f_2\|_{L_{p_2}}. \tag{13}$$

## One more variant of Young's inequality for general spaces of Morrey type

Let

$$\widehat{GM}_{p\theta,w(\cdot)} = GM_{p\theta,w(\cdot)} \cap L_p$$

and

$$\|f\|_{\widehat{GM}_{p\theta,w(\cdot)}} = \max\left\{\|f\|_{GM_{p\theta,w(\cdot)}}, \|f\|_{L_p}\right\},$$

in particular

$$\widehat{M}_p^\lambda = M_p^\lambda \cap L_p$$

and

$$\|f\|_{\widehat{M}_p^\lambda} = \max\left\{\|f\|_{M_p^\lambda}, \|f\|_{L_p}\right\}.$$

**Corollary.** *Under the assumptions of the theorem*

$$\|f_1 * f_2\|_{\widehat{GM}_{p\theta,w(\cdot)}} \leqslant \|f_1\|_{\widehat{GM}_{p_1\theta_1,w_1(\cdot)}} \|f_2\|_{\widehat{GM}_{p_2\theta_2,w_2(\cdot)}}. \qquad (14)$$

If $\theta_1 = \theta_2 = \theta = \infty$, $0 \leqslant \lambda_1 \leqslant \frac{n}{p_1}$, $0 \leqslant \lambda_2 \leqslant \frac{n}{p_2}$, $w_1(r) = r^{-\lambda_1}$ and $w_2(r) = r^{-\lambda_2}$, then inequality (14) takes the form

$$\|f_1 * f_2\|_{\widehat{M}_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \leqslant \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}}. \qquad (15)$$

Note that the spaces$\widehat{M}_p^\lambda$ have the monotonicity property with respect to the parameter$\lambda$: if $0 \leqslant \lambda \leqslant \mu \leqslant \frac{n}{p}$, then

$$\widehat{M}_p^\mu \subset \widehat{M}_p^\lambda$$

and

$$\|f\|_{\widehat{M}_p^\lambda} \leqslant \|f\|_{\widehat{M}_p^\mu}.$$

For this reason there is the “best” among inequalities (15), namely, the inequality

$$\|f_1 * f_2\|_{\widehat{M}_p^{\lambda}} \leqslant \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}}$$

with

$$\lambda = \max\left\{\frac{p_1\lambda_1}{p}, \frac{p_2\lambda_2}{p}\right\}.$$

If $p_1\lambda_1 \geqslant p_2\lambda_2$ it takes the form

$$\|f_1 * f_2\|_{\widehat{M}_p^{\frac{p_1\lambda_1}{p}}} \leqslant \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{L_{p_2}}.$$

## O'Neil's inequality for convolutions of functions

The classical O'Neil's inequality for convolutions of functions for the Lebesgue spaces has the following form:

$$\|f_1 * f_2\|_{L_p} \leqslant c\|f_1\|_{L_{p_1}} \|f_2\|_{L_{p_2,\infty}} \tag{16}$$

for all $f_1 \in L_{p_1}$, $f_2 \in L_{p_2,\infty}$, where

$$1 < p_1, p_2 < p < \infty, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1\,.$$

$c > 0$ depends only on $n, p_1, p_2$ and

$$\|f_2\|_{L_{p_2,\infty}} = \sup_{t>0} t^{\frac{1}{p_2}} f_2^*(t)$$

is the norm in the Lorentz space. (Here $f_2^*$ is the non-increasing rearangement of $f$.)

## Analogues of O'Neil's's inequality for convolutions of functions for Morrey spaces

Counter-examples show that the inequality

$$\|f_1 * f_2\|_{M_p^\lambda} \leqslant c\|f_1\|_{M_{p_1}^\nu}\|f_2\|_{L_{p_2,\infty}},$$

obtained from O'Neil's inequality by replacing $L_p$ by $M_p^\lambda$ and $L_{p_1}$ by $M_{p_1}^\nu$, cannot hold for any

$$1 < p_1, p_2, p < \infty\,, \quad 0 < \nu < \frac{n}{p_1}\,, \quad 0 < \lambda < \frac{n}{p_2}\,.$$

**Theorem 2.** *Let*

$$1 < p_1, p_2 < p < r < \infty\,, \quad 0 < \nu \leqslant \lambda < \frac{n}{p}$$

*and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{\lambda - \nu}{n} = \frac{1}{p} + 1\,. \quad \gamma + \frac{1}{r} = \frac{1}{p_1} - \frac{\nu}{n}\,.$$

*Then there exists $c > 0$ depending only on the above parameters such that*

$$\|f_1 * f_2\|_{M_p^\lambda} \leqslant c\,(\|f_1\|_{M_{p_1}^\nu} + \|f_1\|_{CM_r^\gamma})\|f_2\|_{L_{p_2,\infty}}, \tag{17}$$

*for all $f_1 \in M_{p_1}^\nu \cap CM_r^\gamma$, $f_2 \in L_{p_2,\infty}$, where $CM_r^\gamma$ is the space of all functions Lebesgue measurable on $\mathbb{R}^n$ for which*

$$\|f_1\|_{CM_r^\gamma} = \sup_{t>0} t^\gamma \inf_{x\in\mathbb{R}^n} \|f_1\|_{L_r({}^cB(x,r))} < \infty\,,$$

*where ${}^cB(x,r)$ is the complement of the ball $B(x,r)$.*

**Theorem 3.** *Let*

$$1 < p_1, p_2 < p < \infty\,, \quad 0 < \nu \leqslant \lambda < \frac{n}{p}$$

*and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{\lambda - \nu}{n} = \frac{1}{p} + 1\,. \quad \frac{1}{r} = \frac{1}{p_1} - \frac{\nu}{n}\,.$$

*Then there exists* $c > 0$ *depending only on the above parameters such that*

$$\|f_1 * f_2\|_{M_p^\lambda} \leqslant c\,\|f_1\|_{M_{p_1}^\nu}(\|f_2\|_{L_{p_2,\infty}} + \|f_2\|_{V_{p_2,\infty}^\nu}), \tag{18}$$

*for all* $f_1 \in M_{p_1}^\nu$, $f_2 \in L_{p_2,\infty} \cap V_{p_2,\infty}^\nu$, *where* $V_{p_2,\infty}^\nu$ *is the space of all functions Lebesgue measurable on* $\mathbb{R}^n$ *for which*

$$\|f_2\|_{V_{p_2,\infty}^\nu} = \sup_{t>0} t^{-\nu} \|f_2\|_{L_{p_2,\infty}(B(0,t)\setminus B(0,\frac{t}{2}))} < \infty\,.$$